FOURTH

International Meeting of the APMP

BOOK OF ABSTRACTS

ABEL LASSALLE CASANAVE
GISELE DALVA SECCO (EDS.)

Association for the Philosophy of Mathematical Practice

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UNIVERSIDADE FEDERAL DA BAHIA (UFBA)

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UNIVERSIDADE FEDERAL DA BAHIA BAHIA / SALVADOR - BRASIL

OCTOBER, 23rd - 27th 2017

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SUMÁRIO

EDITOR'S NOTE

The Association for the Philosophy of Mathematical Practice (APMP) is a common forum that articulates and stimulates research in philosophy of mathematics from the perspective of mathematical practice. Created in 2009, the aim of the APMP is to bring together researchers that work on a variety of topics ranging from the way mathematics is done and evaluated to the study of its epistemology, its history, and the educational strategies associated to it. To achieve the goal of creating and maintaining a strong community of researchers, the APMP has organized international meetings. The first Meeting was held in 2010 in Belgium at the Vrije Universiteit Brussel. The second, in 2013, took place in North America, at the University of Illinois, Urbana-Champaign, USA. The third of APMP Meeting, in 2015, returned to Europe. It was held at the Institut Henry Poincaré in Paris, France. Now, in its Fourth edition, the APMP International Meeting will occur in Latin America.

This book presents the programme of the 2017 conference – taking place at the Universidade Federal da Bahia (UFBA), Salvador, Brazil, between the 23rd and 27th of October. Here you will find the abstracts of the invited and accepted papers, as well as the list of all participants.

We would like to express our gratitude and recognize the utmost importance of the support given by various colleagues and institutions that helped the Fourth Meeting to become a reality.

We would like to thank João Carlos Salles, for the indispensable academic and financial support provides as the president of UFBA. In the same vein, we express our gratitude to Olival Freire Jr., Director of Research of UFBA. Special thanks to Waldomiro

Silva Filho (director of the Graduate Program in Philosophy and president of the Brazilian Society for Analytic Philosophy (SBFA)) and also to Luiz Marcio Farias (director of the Graduate Program in Teaching, Philosophy and History of Science). We also thank the Brazilian Society for Logic (SBL) and its president, Cezar Augusto Mortari.

Besides the financial support of UFBA, this event would not have been possible without the encouragement of the Brazilian Federal Agency for Support and Evaluation of Graduate Education (CAPES), and of the National Counsel of Technological and Scientific Development (CNPq). Given the difficult times that Brazil is going through, it is noteworthy that the Meeting was able to secure public investment. We consider it a clear indication of the importance of our area of research and we recognize our responsibility to make the most of this event.

Finally, we would also like to acknowledge the support provided by the APMP, represented by its president, Dirk Schlimm, by all members of the Scientific Committee, coordinated by Marco Panza, and by the members of the Organizing Committee. Last but not least, we express our deepest gratitude to all the colleagues and students who participated in the event. After all, the sharing of our work and our interest in the philosophy of mathematical practice is the very reason why the APMP exists.

The editors.

PROGRAM

• ALL TALKS WILL TAKE PLACE AT HOTEL BARRA SOL •

MONDAY, OCTOBER 23rd	
15:00 – 18:00	**Registration**
Room Farol da Barra	
18:30 – 19:00	**Opening ceremony**
Dirk SCHLIMM – President of APMP Marco PANZA – Chair of the Scientific Committee Abel LASSALLE CASANAVE – Chair of the Organizing Committee João Carlos SALLES – President of the Universidade Federal da Bahia	
19:00 – 20:30	**Opening Lecture** – Chair: Dirk Schlimm
Mathematical impossibility in the social sciences. The history of Arrow's impossibility theorem and its philosophical roots Jesper LÜTZEN	

TUESDAY, OCTOBER 24th	
08:00 – 11:00	**Workshops**
Room 1 – Chair: Luiz Márcio Farias	
The impact of teaching mathematics on the development of mathematical practices Speakers: Gert SCHUBRING / Jorge MOLINA / Tinne HOFF KJELDSEN Discussants: Marcelo AMADEO / Carlos TOMEI	
Room 2 – Chair: Samuel Gomes da Silva	
Beyond Truth and Consistency in Mathematical Practice Speakers: Walter CARNIELLI / Abilio RODRIGUES / Marco RUFFINO Discussants: Emiliano BOCCARDI / Guilherme CARDOSO / Henrique ANTUNES	

14:00 – 17:00	**Talks**
	Room 1 – Chair: José Ferreirós
14:00 – 14:40	*The method behind Poincaré's conventions: structuralism and hypothetical-deductivism* Maria DE PAZ
14: 45 – 15:25	*The Applicability of Mathematics as a Philosophical Problem. Mathematization as Exploration* Michael OTTE / Johannes LENHARD
15: 30 – 16:10	*Material and social conditions for the development of mathematics* Mikkel Willum JOHANSEN / Morten MISFELD
16:15 – 17:00	*Not in the Same River Twice: On the Applicability of Mathematics in Physics* Arezoo ISLAMI
	Room 2 – Chair: Alberto Naibo
14:00 – 14:40	*On Euclidian diagrams and mathematical rigor* Tamires DAL MAGRO
14:45 – 15:25	*From Euclidean Geometry to knots and nets: does Manders' account of Euclidean plane geometry offer a model for the analysis of contemporary mathematical proofs?* Brendan LARVOR
15:30 – 16:10	*Carroll's infinite regress, mathematical understanding, and the act of diagramming* John MUMMA
	17:00 – 18:00 – COFFEE BREAK
18: 00 – 19:30	**Lecture** – Chair: Marco Panza
From counterexamples to examples, or when pathologies become the norm Carmen MARTÍNEZ ADAME	

WEDNESDAY, OCTOBER 25th	
09:00 – 11:00	**Round Table**
Room 1 – Chair: Frank Sautter	
Platonism in mathematical practice Oswaldo CHATEAUBRIAND / Eduardo BARRIO / Marco PANZA	
11:00 – 12:30	**APMP General Assembly**
14:00 – 17:00	**Talks**
Room 1 – Chair: Maria de Paz	
14:00 – 14:40	*Proofs without foundations* Roy WAGNER
14:45 – 15:25	*Visual aspects of scientific models: the case of turbulence* Irina STARIKOVA
15:30 – 16:10	*Diagrams and formulas: on the contents of representations in mathematics* David WASZEK
16:15 – 17:00	*The role of notations in practices of 19th century logic* Dirk SCHLIMM
Room 2 – Chair: Eduardo Giovannini	
14:00 – 14:40	*Carl Snell 'My Revered Teacher': Education, Euclid and System in Frege and his Environment* Jamie TAPPENDEN
14:45 – 15:25	*Are Points (Necessarily) Unextended?* Philp EHRLICH
15:30 – 16:10	*Logic and Proofs in Euclid's Geometry* Alberto NAIBO
16:15 – 17:00	*Euclidean Geometry: Categoricity and the Choice of Logic* John BALDWIN
20:00 – 22:00	**Conference Dinner**

THURSDAY, OCTOBER 26th	
09:00 – 11:00	**Round Table**
Room 1 – Chair: Wagner Sanz	
Formal and informal proofs Luiz Carlos PEREIRA / Jessica CARTER / Max DICKMANN	
14:00 – 17:00	**Talks**
Room 1 – Chair: Davide Crippa	
14:00 – 14:40	*From the comparison of ratios to the comparison of differences* João CORTESE
14:45 – 15:25	*Constructing the Cycloid* Jonhatan ETTEL
15:30 – 16:10	*The Germanic route from negative quantities to natural numbers* Elías FUENTES GUILLÉN
16:15 – 17:00	*What Dedekind's mathematical drafts tell us about the genesis of his lattice theory* Emmylou HAFFNER
Room 2 – Chair: Marco Aurelio Oliveira	
14:00 – 14:40	*Nominalistic content behind the communication problem* Matteo PLEBANI
14:45 – 15:25	*Learning Mathematical Fictions* Pedro CARNÉ
15:30 – 16:10	*Semantic information and the ampliative character of formal knowledge* Bruno R. MENDONÇA
16:15 – 17:00	*Functional and Structural Abstraction. A contribution to concept formation in modern and contemporary mathematics* Bernd BULDT

17:00 – 18:00 – COFFEE BREAK	
18:00 – 19:30	**Lecture**
Room 1 – Chair: Gisele Secco	
Manipulative imagination: from perception and action to mathematics Valeria GIARDINO	

FRIDAY, OCTOBER 27th	
08:00 – 11:00	**Workshops**
Room 1 – Chair: Jessica Carter	
Varieties of visualization in Mathematics Speakers: Silvia DE TOFFOLI / Danielle MACBETH / Javier LEGRIS Discussants: Gisele SECCO / Sandra VISOKOLSKIS / Frank SAUTTER	
Room 2 – Chair: Olival Freire Jr.	
The theories of proportion from Euclid to Hilbert Speakers: Davide CRIPPA / Vincent JULLIEN / Eduardo GIOVANNINI Discussants: Jesper LÜTZEN / Luiz Felipe MIRANDA / André PORTO	
11:30 – 13:00	**Meeting of the APMP Directive Committee**
14:00 – 15:30	**Closing Lecture**
Room 1 – Chair: Abel Lassalle Casanave	
Reducing the Real Numbers: Has the Continuum been tamed? José FERREIRÓS	

APMP 2017

ABSTRACTS
IN ALPHABETICAL ORDER

Zoe ASHTON
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The Normative Role of Audience in Mathematical Proof

While argumentation theory has been adopted in the study of mathematical proofs, much of the research has been focused away from the inclusion of audiences. This lack of focus on audiences stems from a claimed distinction between mathematics and arguments. In this paper, I argue that audiences play a normative role in the judgment of mathematical proofs. By clarifying Perelman and Olbrechts-Tyteca's definition of demonstration it becomes clear that what they banned from their theory of argument was formal derivations, not mathematical proofs as they are practiced. One of the key concepts of Perelmanian argumentation is the ideal of the Universal Audience. In following recent work in argumentation theory, the Universal Audience to embodies standards of reasonableness seen in particular audiences. Part of the reason mathematical proof appears to be free of audiences is that they are actually arguments to the Universal Audience, so the standards of reasonableness seem obviously universal. This is the audience that works behind the scenes, influencing how the mathematician believes a theorem should be proved. This is clear in the proof development stages, when the mathematician argues with two instantiations of the Universal Audience – the mathematician himself and the single interlocutor. In the presentation stages, both the conception of the universal audience and the proof undergo judgment from particular audiences. These particular audiences point out unacceptable gaps in the proof, thereby identifying their particular standards of reasonableness. These newly learned standards are then incorporated into the mathematician's conception of the Universal Audience. By identifying standards of reasonableness with standards of rigor in mathematical proof, we are able to better understand changing rigor. More specifically, it sheds light on the way proofs are treated in their time or field specific contexts.

John BALDWIN
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Euclidean Geometry: Categoricity and the Choice of Logic

Meadows [1] proposes three goals for *Euclidean Geometry: Categoricity and the Choice of Logic:* [a] to demonstrate that there is a unique structure which corresponds to some mathematical intuition or practice; ii) to demonstrate that a theory picks out a unique structure; iii) to classify different types of theory. We expand on his analysis in three ways. We first note that the very notion of categoricity depends on what should be (in our view) a prior notion of isomorphism but that this priority is not at all clear in the history or philosophical literature. Secondly, we argue there may be several specifications of what seems to be a single intuition that lead to different theories and that indeed different logics may be appropriate for formulating these different intuitions. And finally we argue that $L_{\omega 1,\omega}$-categoricity provides a much finer and mathematically meaningful classification for the third goal than 2nd order classification. For the first, note the usual model theorist's notion of isomorphism presupposes a fixed vocabulary τ; each τ-structure is uniquely defined in naive set theory (as in Halmos) and is unique. Since Dedekind, mathematicians have studied the isomorphism type of such structures. But specifying in advance the vocabulary in which the isomorphism is considered (e.g. for Dedekind's natural numbers as a single 'successor function) clarifies the act of 'neglecting special character . . . retaining their indistinguishability . . . taking into account only the relations[1]. Here we will draw on [2], [3], [4], [5]. For the second point we present three different intuitions of 'Euclidean geometry' and present categorical axiomatizations of each; this shows that while categoricity might be a necessary condition for a successful axiomatization it is not sufficient. We call these three geometries: Euclidean, Cartesian, and Hilbertian. The first might more accurately be called constructible; it is the geometry of Euclidean constructions. The names are for convenience and historical accuracy is not asserted. But Descartes was leery of transcendentals – before the term was well-defined. Anachronistically and to save space we describe the geometries by describing in naive set theory the unique models.

Theorem 0.0.1. Each of the following is categorical in $L_{\omega 1,\omega}$. The first two are axiomatized by a single sentence; the third by a family of continuum many sentences.

1. Euclidean Geometry: geometry over the minimal Euclidean field E.
2. Cartesian Geometry: geometry over the the real algebraic numbers[2]; (R^{alg}, +, ×, 0, 1);
3. Hilbertian Geometry: geometry over the complete ordered field (R, +, ×, 0, 1);

Fix the geometric vocabulary to include unary predicates for points and line, a binary incidence relation, and relations for betweenness, segment congruence and triangle congruence. These theories are easily axiomatized

by requiring the first order axioms of Hilbert in each case plus circle-circle intersection in the first.
For categoricity, in the first case specify by an $L_{\omega 1,\omega}$ sentence the quantifier-free diagram of the model, in the second require an Archimedean model of first order geometry over a real closed field, and in the third require all cuts in the rationals to be realized and that the field is Archimedean. These examples are extended by providing first order axioms adding π to 1) and 2) in [6].
Since the basic relations of each model are recursive, it is clear that there are 2nd order axiomatizations of the first two but ones that convincingly represent the minimality in 2nd order rather than sortal logic are harder to come by. These examples show the wide variety of categorical specifications of our first intuitions. As argued in [7], the 'Dedekind' completeness of Hilbert's geometry was intended as a basis for analysis rather than an historically accurate recapitulation of Euclid or Descartes. For the third point, the advantages of the $L_{\omega 1,\omega}$-axiomatization are two-fold. First, they replace the blunderbus of realizing all cuts with the finer geometric structure of (e.g. for 2) asserting all curves (of odd degree) cut the *x*-axis. Secondly, the choice of logic provides a sharper notion of categoricity. The unselectivity of 2nd order categoricity is argued in [8]. (Under V = L, all countable or Borel structures are 2nd order categorical and there are categorical structures of immense cardinality.) But $L_{\omega 1,\omega}$- categorical sentences have a unique minimal (no proper submodel) model and it must be countable. So 'most' countable structures are not $L_{\omega 1,\omega}$-categorical.

Notes:
[1] §73, page 68 of [9].
[2] R^{alg} denotes the real algebraic numbers; this is the maximal field without imaginaries or transcendentals.

References:
[1] Meadows, T. (2013). What can a categoricity theorem tell us? Review of Symbolic Logic, 6:524–544.
[2] Sieg, W.; Morris, R. (2017). Dedekind's structuralism: Creating concepts and deriving theorems. Logic, Philosophy of Mathematics, and their History: (on Morris website).
[3] Reck, E. H. (2003). Dedekind's structuralism: An interpretation and partial defense. Synthese, 137.
[4] Tait, W. (1996). Frege versus Cantor and Dedekind: On the concept of number. In Tait, W., editor, Frege, Russell, Wittgenstein: Essays in Early Analytic Philosophy (in honor of Leonard Linsky), pages 213–248. Open Court Press.
[5] Wilson, M. (1992). The royal road to geometry. Nous, 26:149–180.
[6] Baldwin, J. T. (2016). Axiomatizing changing conceptions of the geometric continuum I. II: Euclid and Hilbert. 55 pages, preprint.

[7] Baldwin, J. (2017). Model Theory and the Philosophy of mathematical practice: Formalization without Foundationalism. Cambridge University Press. to appear.
[8] Baldwin, J. (2014). Completeness and categoricity (in power): Formalization without foundationalism. Bulletin of Symbolic Logic, 20:39–79.
[9] Dedekind, R. (1963). Essays on the Theory of Numbers. Dover. first published by Open Court publications 1901: first German edition 1888.

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Gödel Sentences, Realism and Mathematical Practice

The main goal of Hilbert's program was to formalize all mathematical systems and then prove consistency using only finitist means. As is known, nevertheless, Gödel's incompleteness theorems have significant consequences for Hilbert's elaborate formalism. The theorems seem to show that the necessary metamathematical justification could not be carried out. In particular, Gödel's proof of the Incompleteness Theorems show (i) that Gödel's sentences say of themselves that they are unprovable; and (ii) that they are true provided the theory in question is consistent. How could Gödel's sentences be true if they are not provable? What importance could these truths have for the mathematical practice, if they transcend our capacities of proof? In this talk, I argue - perhaps against Gödel himself- that these results do not imply any ontological point of view. In particular, I reject that these results commit us to some form of mathematical realism (the view that at least some mathematical entities exist objectively, independent of the minds, conventions, and languages of mathematicians).

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Functional and Structural Abstraction. A contribution to concept formation in modern and contemporary mathematics

The traditional approach to concept formation and definition via abstraction presupposes an Aristotelian ontology and its corresponding hierarchy according to which "definitio fit per genus proximum et differentiam specificam." According to this approach, abstraction is tantamount to removing properties and making the corresponding concept less rich; the more abstract a concept is, the fewer content it has. This traditional approach to abstraction and definition does not, however, provide an adequate model for concept formation and definition in mathematics. Actually, it is quite misleading, for a number of reasons. What we need instead of the traditional picture is an account of concept formation and definition that is (1) true to mathematical practice, (2) true to the mathematical experience, and (3) is compatible with insights from cognitive science. We take this to mean in particular that any such account should be informed by historical case studies (to satisfy (1)); that it will result in abstract concepts being oftentimes richer, not poorer in content (in order to meet (2)); requirement (3) needs to be in place for keeping the analysis scientifically sound. In light of the requirements (1)–(3), the paper will identify and discuss various techniques for arriving at mathematically useful abstractions and for defining abstract concepts. While these techniques include familiar examples—for instance, impredicative definitions (e.g., the way Dedekind's characterization of N or R got subsequently emulated) or implicit definitions (basically, modern post-Hilbertian axiomatics)—greater emphasis will be placed on ways to arrive at new, more abstract concepts that are less familiar from recent discussions: parametric abstraction (e.g., the way $Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$ characterizes all conic sections), algebraization of intuition (e.g., intuitive continuity of functions vs. its topological reformulation), and abstraction by embedding in richer theories. (A selection will be made based on the speaking time.) The paper takes the discussion in [1] as its starting point.

References:

[1] Buldt, Bernd; Schlimm, Dirk. "Loss of vision: How mathematics turned blind while it learned to see more clearly," in: Philosophy of Mathematics: Sociological Aspects and Mathematical Practice, ed. by Benedikt Löwe and Thomas Müller, London: College Publications (2010), pp. 87–106.

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Learning Mathematical Fictions

In this talk, I intend to analyze Azzouni's notion of *mathematical fictions* advanced by him on his homonymous paper. My main goal is to derive from this notion some claims regarding mathematical practice. More specifically, I shall address Azzouni's emphasis on deductive tractability of mathematics as a property closely connected with the allegation that mathematical terms do not refer in order to understand how it would be possible to learn mathematics. That is, according to Azzouni's theory, what do we learn when we learn mathematics? For unpacking this question, it is fundamental to grasp in which way Azzouni deals with three different and complementary claims: (i) that a statement that is true (or false) about something is about "some *one* thing;" (ii) that mathematics is empirically valuable; and (iii) that deductive reasoning seems to yield necessary truths. By grasping Azzouni's arguments targeted at each one of these claims it is possible to look into his deflationary nominalism, according to which "it is perfectly consistent to insist that mathematical theories are indispensable to science, to assert that mathematical and scientific theories are true, and to deny that mathematical objects exist." (Bueno 2013) The idea that fictions are only connected with entertainment is challenged by Azzouni throughout his paper, and by the end of it he maintains that the philosophical puzzles raised by fictions are indeed very deep, given that their internal mechanisms are related to intricate issues, such as how reasoning actually works and how our cognitive faculties enable us to engage in imaginative thinking (and talking).

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Contradictory and inconsistent sets, Zermelian set theory, and forcing

Since the discovery of the paradoxes, research in contemporary set theory has focused on attempts to rescue Cantor-Frege's naive theory from triviality. One way to escape such deductive trivialism is to weaken the underlying logic and to allow for contradictory sets, possibly taking profit of their exotic character. This move would reconcile set theory with Cantor's liberal approach: although Cantor never accepted a contradiction as a meaningful mathematical tool, he tolerated the existence of inconsistent totalities and even reasoned with them. What we call *Zermelian paraconsistent set theories* in [4] are systems, as the one presented in [1], that accept the axiomatization that Zermelo proposed in his 1908 paper for the development of set theory, but using an underlying paraconsistent logic. Defenders of this line are B. Löwe, S. Tarafder, W. A. Carnielli and M. E. Coniglio among others. An alternative position is represented by the *Cantorian paraconsistent set theories*, which aims at formalizing set theory just by means of Extensionality and unrestricted Comprehension. Typical proponents of this line are R. Routley, G.Priest, R. Brady and Z. Weber. George Cantor's seminal intuitions on sets can be vindicated in the light of paraconsistency, specially by employing the logics of formal inconsistency. The Zermelian approach in [1] (see also [2], chapter 8) intends to propose an axiomatic system as close as possible to **ZFC**, but where the logical connectives receive a paraconsistent interpretation. The primary intuition is to assume that not only theories can be taken to be consistent or inconsistent, but also that sentences and sets themselves can be thought to be consistent or inconsistent, expressed by a new operator ◦. The basis for new paraconsistent set-theories such as **ZFmbc** and **ZFCil** are established from this perspective, and their nontriviality is proved provided that **ZF** itself is consistent. This shows that the new operator ◦, which permits one to separate contradictions from inconsistency and avoids deductive triviality, is a natural and effective tool for expressing contradictory objects in mathematical practice, in quite a similar fashion to the way complex numbers behave. A further step is then the formidable question of proving the independence of CH in a paraconsistent setting, a task that claims for methods for constructing models of paraconsistent set theory, as in [3]. In order to provide the consistency (or the independence) of CH with respect to a paraconsistent set theory, a notion of forcing and inner model with a paraconsistent nature are needed, a problem discussed (and partially solved) in [4].

References:
[1] Carnielli, W. A.; Coniglio, M. E. Paraconsistent set theory by predicating on consistency. Journal of Logic and Computation, 26(1):97–116, 2016.
[2] Carnielli, W. A.; Coniglio, M. E. Paraconsistent Logic: Consistency, Contradiction and Negation. Series: Logic, Epistemology, and the Unity of Science Springer, 2016 http://www.springer.com/la/book/9783319332031
[3] Löwe, B.; Tarafder, S. Generalized algebra-valued models of set theory. Review of Symbolic Logic, 8(1):192–205, 2015
[4] Carnielli, W. A.; Coniglio, M. E.; Venturi, G. Paraconsistent set theory and forcing Manuscript.

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Proofs in contemporary mathematical practice

My presentation will focus on proofs in contemporary mathematics and so informal proofs. Moreover I will consider the use of diagrams in such proofs. By presenting some of the roles diagrams play in proofs my hope is to stimulate a discussion about formal versus informal proofs in mathematics. In the literature one sometimes find statements claiming that diagrams should play no justificatory role in mathematical proofs referring, for example, to quotes of Hilbert and Pasch from the beginning of the 20th century. Considering proofs in contemporary mathematics journals, however, one may find a variety of diagrams referred to. In the first part of the talk I will show some examples of diagrams employed in analysis and thus documenting this claim. Second I will consider some of the roles these diagrams play. In this part I will mainly focus on a case study from the area of C*-algebras where certain diagrams, so called directed graphs, are used to represent and generate these algebras. In addition to being much simpler objects to study, the advantage of these graphs is that it is possible to define certain invariants of the C*-algebras (called K-groups) directly from the graphs – and that the definition of these invariants is extremely simple. In this sense I will explain that these graphs act as mediating object between the C*-algebras and their invariants. Furthermore I will argue that this function is made possible because these graphs – or diagrams – can be taken to represent in two different ways. In other words they represent metaphorically, that is, because of some given rules. Reading a particular graph according to one set of rules one obtain the relations generating a particular C*-algebra. Reading it following a different set of definitions one obtains the invariants. Another important feature of the graphs is that they are objects that can be manipulated and so experimented on. Adding vertices or edges to a graph (in a controlled way) gives rise to different algebras and their corresponding invariants. In this way it is possible, for example, if given a certain set of invariants to re-construct the particular graph and so the C*-algebra that has these invariants. In the last part of the talk I will indicate that this role of a representation, that is, that it can be manipulated on and that manipulations respect certain relations, is not exclusive for diagrams but occurs also in mathematical proofs in general and that this function is (in)valuable in mathematics.

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The ontology of mathematical practice revisited

As Bernays maintained in his famous 1935 article "On Platonism in Mathematics", in mathematical practice objects, functions, relations, properties, structures, etc. are treated as entities that exist independently of our discourse and of our constructions. Bernays maintains that this form of Platonism is essentially a manner of speaking, which does not involve a commitment to a strict form of Platonism. In a talk at the 2012 Conesul meeting "The ontology of mathematical practice" (published in *Notae Philosophicae Scientiae Formalis* vol. 1, n. 1) I defended Bernays' position and proposed a more systematic formulation for this form of Platonism, combining it with ideas of Frege and of Gödel. In the round table discussion, I intend to elaborate this theme.

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From the comparison of ratios to the comparison of differences

Proportion theory for magnitudes appears in its classical form in book V of Euclid's *Elements*. It presents conditions about what kind of relations between two magnitudes can constitute a ratio (logos) (by the Archimedean Property or Eudoxus' Axiom, given in definition 4). Besides, four magnitudes are said to be proportional if they are in the same ratio, the first to the second and the third to the forth (definitions 5 and 6). Besides, according to definition 7 a ratio can be bigger than another, constituting another aspect of the comparison of ratios. In Seventeenth-century mathematics, proportion theory was well known and widely used. At the same time, new theories were been developed at this time, and new questions about the comparability of magnitudes appeared – in particular, with the methods of indivisibles. One of the biggest discussions brought by indivisibles was whether they were homogeneous or heterogeneous to their corresponding magnitudes – a problem directly related to the existence of ratios. Besides, the comparability of ratios was of greatest importance for demonstrations in which an infinite (or indefinite) number of indivisibles are considered: would like to discuss an important aspect of Blaise Pascal's treatment of differences (errors) between sums of indivisibles and the corresponding magnitudes, developed in his Lettres de A. Dettonville. If on the one hand Pascal himself declares that his method is equivalent to the "Method of the Ancients", on the other hand he makes an important contribution as he takes the differences themselves as elements that can be in some way quantified and compared. This new treatment of error by Pascal allows a comparison between two kinds of "indivisibles": the small portions of the magnitude considered, and the differences. These comparisons appear in a kind of proof technique that shall be considered in the light of the delta-epsilon modern definition of limit, developed by Cauchy, Bolzano and Weierstrass. Even if Breger (2008) claims that "the connection between infinitesimals and what we now call epsilontics was obvious enough for 17th-century mathematicians", a precise analysis of Pascal's contribution and is specificity is necessary, since Whiteside (1961) recognised the originality of Pascal's usage of differences, but did not give an exhaustive philosophical analysis of it.

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The Foundational Role of the Theory of Proportions in 17th Century: Viète and Descartes

One of the most interesting aspects of the history of the theory of proportion in 17th century is its intimate relationship with symbolic algebra, namely with the study of equations. This relation regarded both arithmetic and geometry. In this talk I will discuss the role played by Euclid's theory of proportion in two outstanding geometrical works, and particularly in Viète's *Introduction to the Analytic Art* (1591) and in Descartes' *Geometry* (1637). My thesis is the following: Viète's and Descartes' works can be interpreted as two efforts to establish algebra, which had grown up as a cluster of practices, as a science by grounding its symbolic manipulation on Euclid's geometry. Euclid's theory of proportions played a crucial role in these foundational tasks. I do not claim that this was the main goal of Viète's and Descartes' mathematical programmes, but that it was a relevant consequence for both of them. The ways and the extent to which this project succeeded will be discussed in this talk. Attempts to employ geometrical considerations to verify algorithms for the solution of algebraic equations up to the 3rd degree were common among Renaissance algebraists (Tartaglia, Cardano, Bombelli). Geometry was to be preferred to arithmetic because of its greater generality. Indeed by "geometrical considerations" I refer to the possibility of grounding every algebraic manipulation into a corresponding geometric construction, whose correctness is ultimately proven on the basis of Euclid's *Elements*. The correspondence between algebra and geometry was based on the so-called principle of homogeneity, namely on the correspondence which associates x to a segment, x^2 to a square and x^3 to a cube. This correspondence was broken for equations of degree higher than the third, due to the lack of fourth (and higher) dimensional geometric objects. However, several answers were found between the second half of 16th and the first half of 17th century in order to overcome this difficulty and ground manipulations with higher algebraic quantities on geometry. Viète's idea, inspired to Diophantus, consists in basing his symbolic algebra, or *logistica speciosa,* on Euclid's theory of proportions. If this programme were feasible, then any algebraic equation of any degree could be well-grounded into Euclid's *Elements*, which stood as the model of rigour. However, this possibility could not be taken for granted, and one of the main tasks of Viète's inquiry was indeed the study of how algebraic equations could be associated to proportions (*De recognitione aequationum*, 1615). This seems to have been for Viète a purely theoretical, or foundational task, independent from pratical applications of symbolic algebra. Euclid's theory of proportion also plays a central role in Descartes' reflection on mathematics in the *Géométrie* (1637), but in an utterly different way with respect to Viète. In particular, the theory of proportion has an operational function in the *Geometry*, because it grounds a geometrical calculus or an "algebra of segments". By virtue of this role, the connection between algebra and the theory of proportions is crucially different in Viète with respect to in Descartes: while one of the main tasks which led the inquiry of the former was to interpret equations in terms of proportions, for the latter equations are from the start, by virtue of the geometric interpretation of arithmetic operations, compact notations for proportions between segments. These differences would be stressed by the first commentators of the *Geometry*, who saw Descartes' geometrical calculus also, but not only, as a way to ground symbolic algebra on the solid basis of Euclidean geometry in a more convenient and consistent way than Viète's.

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On Euclidean Diagrams and Mathematical Rigor

In this work we investigate the consequences of defending the legitimacy of diagram use in Euclidean proofs (Manders) i.e., of endorsing the thesis that diagrams can be used in proper demonstrations without compromising their rigor. Thus, we are concerned with a conception of (mathematical) proof that is broad enough to encompass diagrammatical information. Such a conception stands in shark contrast to the conception of proof put forward in the late 19th and early 20th centuries, according to which proofs are, by definition, sequences of sentences such that their demonstratives steps are either axioms or follow from them by mean of inference rules. That conception quickly became the orthodoxy in metamathematical thinking and the epistemic properties of diagrams fell into discredit. We intend to assess a proposal under development by Ferreirós and also, independently, by Lassale Casanave & Panza, according to which *Elements,* as much as any other robust mathematical text, should be understood as a mathematical treatise, i.e. not only as a collection of solved problems and demonstrated theorems, but also as a work exposing a *theory* and a manner of doing mathematics inside that theory. In another words, the authors defend that an adequate analysis of mathematics should take into account the different mathematical practices that exist (or have existed), their symbolic frameworks (be it formula-, diagram-based etc.) as well as the particular objectives defining the distinct mathematical communities' employment of these frameworks. From this perspective, both Euclid's plane geometry and its formal-axiomatic counterparts earn the merit of being considered full-blown mathematical theories in their own right, and we become able to rebuke the claim that the latter is a perfection of the former and the claim that "Hilbert 'perfected' Euclid's axiomatization". On the basis of those considerations, we explore the idea that a conception of mathematical demonstration or proof should be extended so as to be sensitive to the distinct symbolic frameworks and manner of doing mathematics that constitute distinct mathematical practices.

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The double origin of Poincaré conventionalism: methodological structuralism and hypothetical-deductive method

The origins of Poincaré's conventionalism have always been situated in problems about the status of certain scientific principles caused by the development of new scientific theories. Thus, it is typically a philosophical position emerged from scientific practice. For example, regarding geometrical conventionalism, it was the existence and consistency of nonEuclidean geometries what prompted the development of a philosophical position that could account for the status of the axioms of geometry without engaging in a discussion about their truth. Similarly, conventionalism in physics and mechanics was provoked by the discussion of the status of its fundamental principles, starting with the Newtonian laws of motion. Conventions have been the object of several contemporary philosophical debates, e. g. concerning their agreement with modern science (for example, with general relativity), their rigidity, flexibility, or constitutivity in scientific theories, their relation to realistic or antirealistic positions, and so on. Here I would like to consider conventionalism as a philosophical position originated from two specific methodologies proper to modern mathematics and the modern natural sciences: methodological structuralism and hypothetical-deductive method – thus, as a philosophical position which emerged from a way (or rather, two ways) of doing science. What is more important, I will try to show that these two methods are connected in both disciplines, geometry and physics. First, Poincaré's claim that geometry is the study of the formal properties of a certain continuous group is typically a structuralist claim. And we will show that geometrical conventionalism could be understood as a consequence of this way of doing and understanding geometry. Second, Poincaré's defense of a 'physics of principles' where the conventional status of the principles is made explicit can be linked to a movement of abstraction in 19th century physics that is similar to the conceptual approach in mathematics (characteristic of the structuralist position). Third, Poincaré's discussion of the status of geometrical axioms and his reading of them as conventions is related to Riemann's understanding of geometrical axioms as hypotheses. The use of the term hypothesis implies the non-certain (hypothetical) character of the axioms, as well as the possibility of choosing different sets of them. Fourth, the treatment of physical principles as conventions also stresses their hypothetical character, since they cannot be established as true or false and alternatives are possible. Also, hypothetical-deductive method emerges by the mid-19th century and is visible in Riemann's manuscripts on mechanics and physics. In fact, the availability of alternatives and the possibility of choosing is what connect conventions and hypotheses. The words 'convention' and 'hypothesis' have to do with methodological flexibility, that is, with the idea of having different conceptual possibilities, and the conceptual approach is typically represented by the structuralist position.

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What is a Mathematical Diagram?

In the literature on visualization in mathematics, terms such as *diagram* and *graphic representation* have been used in different and even contradictory ways. In this talk, I propose a new working vocabulary to disambiguate these and other related terms, and distinguish different types of notations and visualizations in mathematics. My focus will be on external representations: I will not consider mental visualizations. I will first define different types of representations and then see how they combine forming systems of representations (i.e. languages or mathematical notations). I will start by considering *figures* in mathematics, that is, two-dimensional displays of mathematical content exploiting in a non-trivial way the space of the page, i.e. not only constituted by symbols in sequence. Of course, also symbols in sequence exploit two-dimensionality, but they do so trivially, in at least two ways: the symbols themselves are two-dimensional and sequences are divided into different lines. I will distinguish between two types of figures: *illustrations* and *diagrams.* On the one hand, illustrations are figures that represent in a non-constrained way mathematical content. Examples of illustrations are computer-generated images of topological manifolds and illustrations in analysis (e.g. the one of the Intermediate Value Theorem.) On the other hand, diagrams are more constrained representations. They are two-dimensional figures that present rigid constraints on their shape and interpretations and that allow for a specific type of mathematical reasoning on them. A mathematical diagram presents tight constraint on: i) its formal properties, ii) its interpretations, and iii) its possible manipulations. Thanks to these constraints, mathematical diagrams, like mathematical one-dimensional representations, do not only represent content statically, but can be manipulated in specific ways to support mathematical inferences. In this sense, they are epistemic tools that allow mathematicians to reason. According to the proposed definition, mathematical diagrams are contrasted both to linear, or one-dimensional, mathematical representations, and to illustrations, that is, two-dimensional unconstrained representations. Note that the boundaries between these categories can be fuzzy and there are hybrid cases. The use of diagrams in mathematics involves spatial thinking, but not necessarily geometric thinking; the reasoning could be entirely combinatorial, as in the case of diagrams in graph theory. In order to isolate the subcategory of diagrams which allow for geometric (or better, topological) type of reasoning, I will define *graphic representations*. Diagrams that are graphic exploit topological notions of the plane in which they are inscribed, such as the Jordan curve theorem. Examples are knot diagrams and Euclidean diagrams, while example of diagrams that are not graphic are oriented graphs and commutative diagrams. Proposing a working taxonomy for visualization in mathematics, I aim at shedding light on the nature of different kinds of mathematical diagrams and on the basis of the possibility of using them as actual epistemic tools to reason in mathematics.

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Formal vs informal proofs in mathematics

My presentation will consist on a reflection on informal proofs, as done by working mathematicians, versus the standard of rigor required in formal

mathematics, and the implications of the gap between them. To some extent it will be based on personal experience.
The « no man's land » between formal and informal proofs has delicate implications for (at least):
(1) The transmission of mathematical knowledge at both the levels of teaching and of creation.
(2) The writing of mathematical texts, both books and research papers.
In fact, the chasm deepens as the mathematical level increases.
At the time of creation (« doing » maths) a decisive component of a (necessarily informal) new proof is intuition. This, in turn, depends on prior knowledge (and/or habit) built along a formative period, or a life-long mathematical practice or, more prosaically, upon taste and will. Taking into account that, as a result of sheer accumulation, any single mathematician is familiar with only an infinitesimal fraction of the corpus of the mathematics of her/his time, the difficulty in appreciation of informality --on which much of the *understanding* of mathematics rests-- becomes a stumbling block for mathematical communication even among colleagues of the same level and practising closely related disciplines. Just a pedestrian example: anyone who has written research papers has not failed to have the feeling that the reviewer of his paper « has not *really* understood it ». This is not always due to the referee's ill intentions, but frequently to a difference of appreciation of the value, and the difficulties involved in the creation of the piece of mathematics under examination, due to a difference in formation --hence of perception and focus-- between author and reviewer. Differences of this type are subtle, but permeate the trade.

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Are Points (Necessarily) Unextended?

Ever since Euclid defined a point as "that which has no part" it has been widely assumed that points are necessarily unextended. It has also been assumed that, analytically speaking, this is equivalent to saying that points or, more properly speaking, degenerate segments—i.e. segments containing a single point—have length zero. In our talk we will challenge these assumptions. We will argue that neither degenerate segments having null lengths nor points satisfying the axioms of Euclidean geometry implies that points lack extension. To make our case, we will provide models of ordinary Euclidean geometry where the points are extended despite the fact that the corresponding degenerate segments have null lengths, as is required by the geometric axioms. The first model will be used to illustrate the fact that points can be quite large—indeed, as large as all of Newtonian space—and the other models will be used to draw attention to other philosophically pregnant mathematical facts that have heretofore been little appreciated, including some regarding the representation of physical space. Among the mathematico-philosophical conclusions that will ensue from the above talk are the following three. (i) Whereas the notions of length, area and volume measure were introduced to quantify our pre- analytic notions of 1-dimensional, 2-dimensional and 3-dimensional spatial extension, the relation between the standard geometrical notions and the pre-analytic, metageometric/metaphysical notions are not quite what has been assumed. Indeed, what our models illustrate is that, it is merely the infinitesimalness of degenerate segments relative to their non-degenerate counterparts, rather than the absence of extension of points, that is implied both by the axioms of geometry and these segments null lengths. (ii) As (i) suggests, the real number zero functions quite differently as a cardinal number than as a measure number in the system of reals. So, for example, whereas a set containing zero members has no member at all, an event having probability zero may very well transpire, and perhaps more surprisingly still, a segment having length zero may contain a point encompassing all of Newtonian space. (iii) Physicists and philosophers alike need to be more cautious in the claims they often make about how empirical data determines the geometrical structure of space. Indeed, without additional geometrical assumptions, which their writings typically show no cognizance of, the evidence that is often appealed to is compatible with an array of distinct geometrical spaces. For example, as three of our models collectively demonstrate, even if one could have good experimental evidence that the sum of the interior angles of some (any) triangle in the plane is 180°, this would be entirely compatible with (a) an Archimedean Euclidean space, (b) a non-Archimedean Euclidean space, (c) a nonArchimedean hyperbolic space, (d) a non-Archimedean elliptic space, (e) a nonArchimedean semi-hyperbolic space that is not hyperbolic, (f) a semi-elliptic space. We believe this has interesting implications for traditional philosophical disputes regarding the epistemology of geometry including the classical dispute between empiricist and conventionalist philosophies of geometry. We will conclude our talk by drawing attention to the latter.

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A Priori Concepts in Euclidean Proof

Famously, Euclid's proof of Proposition 1.1. in the Elements is taken to contain a "gap": nothing in Euclid's axioms guarantees the existence of the intersection point of the two circles he constructs in the course of the proof, so the proof is not (according to the standard criticism) valid. If we take Euclidean geometry to be a system of deductive logic, this gap requires plugging with an explicit formal definition of continuity – a definition that was worked out only in the nineteenth century, over two thousand years after Euclid's time. And yet, working without any such formal definition, Euclid produced only correct proofs: all of his theorems are provable in the suitably "completed" systems of Hilbert and Tarski. This suggests that Euclid was not merely working within an incomplete system of logical deduction (where his ability to produce only correct proofs would seem quite miraculous), but rather was engaged in an entirely different kind of practice – a form of specifically geometrical reasoning, in which the content of concepts like *line* and *circle*, not just the form of the axioms in which those concepts feature, plays a crucial role. In this paper, I argue that the concepts in question—the concepts whose contents fill the "gaps" in the formal structure of Euclid's proofs—cannot plausibly be taken to be empirical. For the proof of *Elements* 1.1. turns on the distinction between genuine continuity and mere denseness: the proof fails, for instance, if the circles are taken to be constructed in the merely-dense rational plane.[1] And the distinction in question—the distinction between a construction in the continuous real plane, where the circles do indeed intersect, and a construction in the rational plane, where the dense (but not continuous) circles fail to intersect—is not something that could ever be derived from experience. The two versions of the construction would be visually identically, no matter how carefully, or at what magnification, the circles were viewed; so the needed distinction cannot be an experiential one. Building off of this point, I critically assess three attempts—due to Strawson, Manders, and Giaquinto—to analyze Euclidean proof as deriving from visual reasoning. In order to assess these accounts, I propose a criterion for deciding whether a given mathematical concept is visually derivable, based on the idea that we can utilize a kind of "limit procedure" to extract such concepts from experience. Using this criterion, I show that a concept of denseness is, in an important sense, derivable from experience; but that the further distinction between mere denseness and genuine continuity—the distinction needed to ground Euclid's proofs—is not. Thus, I argue, in order to account for the success of Euclid's system, we must acknowledge the existence of a special form of a priori geometrical reasoning at work in the practice of Euclidean proof – a kind of a priori thought distinct from purely formal, deductive reasoning, which utilizes a primitive set of contentful, non-visual spatial concepts.

1. See: Friedman, Michael. Kant and the Exact Sciences (Chapter 1). Cambridge: Harvard University Press, 1992. Strictly speaking, the proof of Elements 1.1. does not require the full continuity of the real plane. All the proofs found in Euclid's Elements can be performed assuming only a plane based on the so-called "Euclidean extension" of the rationals. Such a plane contains curves that are not continuous, but it will suffice for all of Euclid's existence claims about points. The important point, though, is that there are some dense-but-not-continuous curves—those of the rational plane—that would fail to intersect as assumed in Euclid 1.1.

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Constructing the Cycloid

The seventeenth century marked the introduction of a host of new curves into planar geometry. Of these the cycloid stands among the most famous and most widely studied. It also lies at the center of a number of philosophical and methodological debates. First, it is not definable by a finite equation in rectangular coordinates, and hence according to the criterion advanced by Descartes, is ``mechanical'' rather than ``geometrical'', and not subject to exact knowledge. Second, it was proposed by father Mersenne as a test case which could display the power of the method of indivisibles, a suggestion carried out notably by Roberval and Pascal, and hence the curve is also bound up with questions about the power, fruitfulness and certainty of the method, itself beset by controversy. Finally, it figured in important physical and technological applications, most notably in Huygens' construction of the pendulum clock, where metal plates bent into the shape of the cycloid restrict the pendulum cord, creating a simple harmonic oscillator. The means by which the construction of the curve is effected differ markedly from case to case. The curve is most straightforwardly described as that generated by a point on a circle rolling along a straight line so that it makes one revolution in the time it traverses a lengthy equal to its circumference. In practice descriptions of this form are sometimes supplemented by alternative specifications, as in Roberval's resolution of the defining motion into indivisible elements. In other cases, entirely different modes of construction are proposed, as in Huygens use of a pointwise approximation. We examine a number of constructions of the curve in the works of Roberval, Pascal, Huygens, Newton and others, and note that they support alternative representations, each of which captures some virtues at the sacrifice of others, where these virtues include exactness, conciseness of demonstration, and applicability in calculation and mechanical construction.

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Structuralism in mathematics: a conceptualist approach

In this talk I shall discuss the approach to structuralism that has been proposed in recent years by S. Feferman, which he called 'conceptual structuralism', in connection with my own conceptualistic epistemology. Feferman's contrasts with other forms of structuralism, such as Shapiro's ante rem, for its avowedly non-realistic stance. Yet a discussion of this topic shall lead us to formulate a form of lightweight realism that avoids the pitfalls of more extreme standpoints. In this light, I shall argue that the crucial question is not the metaphysics of mathematical objects, but the question of objectivity of mathematical knowledge -- and here again I agree with Feferman. The talk will conclude with a review of my arguments for objectivity, presented in a recent book (2016), and some finishing remarks.

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The Germanic route from negative quantities to natural numbers

From Kästner's designation of the arithmetic of –positive– whole numbers as *natürliche Arithmetik* (Kästner, 1758) to Bolzano's commentary on those numbers as *der sogenannten natürlichen* (Bolzano, 1851) almost a century passed, during which a change in the notions of number and quantity took place among Germanic mathematicians. As evidenced by the fact that while in the entry "Nombre" in the *Encyclopédie* it was stated that –positive– whole numbers were "also known as natural numbers" (*Encyclopédie*, 1765), in the entry "Zahl" in Klügel's *Mathematisches Wörterbuch* that alternative appellation was not employed (Klügel et al., 1831), cultural difference was important in the emergence of the notion of natural numbers. My talk will focus on procedural and conceptual changes in the works of the main Germanic mathematical authors of the second half of the 18th century that show such emergence. For them, on the one hand, mathematics was the science of quantity, and "quantity", if taken rigorously, could not be applied to 0; this explains their procedures to introduce infinitely small quantities within the framework of analysis and their reluctance towards these quantities as developed by Euler and others. On the other hand, for them arithmetic considered quantity in terms of numbers, and "numbers", in the strict sense of the idea, were only the –positive– whole numbers, from which proper-fractions of-numbers could be formed; this explains why as negative numbers were gradually accepted it became less unusual to use the denomination of "natural numbers" to refer to those whose positivity only arose once the others were introduced. I will defend that: a) The emergence of the notion of natural numbers among those mathematicians, as well as their insistence on "negatively expressed quantities" as the proper designation of the so-called negative quantities, which in turn could be considered numerically and thus give rise to negative numbers, highlight the exisiting Germanic reluctance towards the status as numbers of the latter. b) This was intertwined with the Germanic tension within the framework of analysis to incorporate some of the new "foreign" developments but reject some others, as evidenced, for example, by Karsten's procedures when he introduced the "true and correct" infinitely large quantities ($n/0 = \infty$) and at the same time discarded the infinitely small quantities ($n/\infty = 0$) for not being quantities at all. c) The involvement of those authors in these processes shows that their works cannot even in their beginnings be considered as "more popular and readable" versions of Wolff´s work (e.g. Bullynck, 2006), though they were not entirely different either: the incorporation and acceptance of some of those concepts and procedures did not occur in their first works (e.g. Schubring, 2005) but over the years. That way, I intend to contribute to a better understanding of the process that led to the modification of the notions of quantity and number among those Germanic mathematicians and, in doing so, correct some historiographical misconceptions on the subject, common even among contemporary authors.

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Manipulative imagination: from perception and action to mathematics

In the first part of my talk, I will briefly present previous work with De Toffoli on the practice of topology. We have proposed that topologists, in order to become experts, have to learn how to use what we have defined as *manipulative imagination*. Such a form of imagination is central to many areas of topology, for example knot theory (De Toffoli & Giardino, 2014), low-dimensional topology (De Toffoli & Giardino, 2015) and braid theory (De Toffoli & Giardino, 2016). To clarify, in order to follow the proofs, topologists have to envisage transformations *of* and *on* the diagrams. Their interaction with the representations is therefore essential: the figures are not static, but have to be used *dynamically* so as to trigger a form of imagination that allows them acting on them and drawing inferences accordingly. Representations are thus cognitive tools whose functioning depends in part from pre-existing cognitive abilities and in part from specific training. If manipulative imagination exists, and possibly it is used also in other areas of the sciences, what kind of imagination is it? In the second part of my talk, I will refer to the notion of imagination as "make-believe" as proposed by Walton (1990) to give an account of the role of cognitive tools in mathematics as *props*. To better specify my claim, I will also rely on the notion of "affordance" as proposed by Gibson (1979) and discuss how it can be extended from concrete objects to representations.

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'Harmonizing Euclidean geometry': Hilbert and the theory of proportion

The aim of this talk is to provide a historically sensitive discussion of Hilbert's reconstruction of the theory of proportion in his groundbreaking monograph *Foundations of geometry* (1899). It will be argued that Hilbert bestowed to this reconstruction a crucial epistemological and methodological significance. On the one hand, the theory of proportion was for Hilbert one of the central parts of elementary geometry that called more urgently for a new solid foundation. On the other hand, an adequate 'purely geometrical' grounding for the notion of proportionality was essential for his main aim of providing an *independent basis* for geometry, since this notion was necessary for the reconstruction of other important parts of elementary geometry, such as the theories of similar triangles and plane area. The presentation will be structured in two main parts. In the first part, I will present and analyze some critical comments formulated by Hilbert to Euclid's theory of proportion developed in Book V of the *Elements*. These critical remarks consist in pointing out that the Euclidean theory does not have a 'purely geometrical' character, since Euclid never explains what it means *geometrically* for two pairs of geometrical elements – *e.g.*, line segments – to be *proportional.* Moreover, Hilbert observes that the Euclidean theory of proportion is grounded on an arithmetical basis, which can be noted in the fact that the definition of proportionality provided by Euclid requires the validity of a continuity principle such as the axiom of Archimedes. Then, I will argue that Hilbert's objections to Euclid's theory of proportion and similar triangles consisted not only in pointing out the existence of implicit assumptions, but also in raising very explicit purity concerns. In the second part of the presentation, I will expound briefly the technical content of Hilbert's theory of proportion, that is, his definition of proportionality on the basis of the arithmetic of segments [*Streckenrechnung*]. Hilbert showed that, once the operations of sum and product of line segments have been defined in an adequate and purely geometrical way, it is possible to use the classical theorems of Desargues and Pascal to prove that these operations satisfy all the properties of an ordered field. This purely geometrical construction of a set of segments, which satisfies all the properties of an ordered field, allowed him to reconstruct the classical Euclidean theory of proportions and similar triangles, to which he finally resorted to perform an *internal* arithmetization of geometry. Hence, Hilbert produced a unification of two theories, which before were grounded on different foundations, giving at the same time a new answer to the problem of the role of numbers in geometry.

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What Dedekind's mathematical drafts tell us about the genesis of his lattice theory

When Richard Dedekind introduces the notions of module and ideal in his famous 1871 Supplement X to Lejeune-Dirichlet's *Vorlesungen über Zahlentheorie*, he also defines notions of divisibility (e.g. a module a is divisible by a module b if $a \rightarrow b$) and related arithmetical notions for modules and ideals (e.g. LCM and GCD of modules or of ideals). Without specific notation for these new concepts and methods, Dedekind proves the general validity of the unique factorization theorem for algebraic number fields. In 1877, in *Über die Anzahl der Ideal-Klassen in den verschiedenen Ordnungen eines endlichen Körpers*, the introduction of notations for divisibility, LCMs and GCDs of modules allows Dedekind to state new theorems, which are now recognized as the modular laws in lattice theory. Observing the dualism displayed by the theorems, Dedekind pursues his investigations on the matter, and is led to the introduction of the notion of *Dualgruppe* (equivalent to our modern-day lattice). The notion is introduced in the 1894 version of Dirichlet's *Vorlesungen* (under the name *Modulgruppe*), but Dedekind only exposes his theory of Dualgruppe in 1897 (*Über Zerlegungen von Zahlen durch ihre größten gemeinsamen Teiler*) and 1900 (*Über die von drei Moduln erzeugte Dualgruppe*). It was, in his words, obtained "not without great effort" (Dedekind 1897, 113) and indeed after two decades of work. In this talk, I propose to study the long genesis of Dedekind's *Dualgruppe* with the help of drafts kept in his *Nachlass*. Dedekind's perfectionism and great attention to details is well-known, and it is not surprising that he kept this work under wraps for such a long time. Diving into Dedekind's *Nachlass*, one can find an impressive quantity of notes and computations around the arithmetical operations for modules and ideals leading to the slow, progressive elaboration of the notion of *Dualgruppe*. I will show how Dedekind gradually builds his Dualgruppe theory through many layers of computations – often repeated in slight variations and attempted generalization – by exploring the various possibilities, laws, etc. offered by these arithmetical operations and by the dualism of the theorems with series of examples, tables, and calculations. In this procedure, he looks for remarkable properties of these operations, properties susceptible to be applied elsewhere, generally valid properties, and he attempts to identify which properties should be considered as "fundamental" ones. These computations and the stepwise generalization of the concept largely disappear from the published exposition of the theory, which appears as very general and abstract. Insofar as these drafts are working tools for Dedekind, by studying the concealed strates of mathematics they contain, I wish to reveal and clarify the preliminary and intermediary states of the mathematical research, the essential yet hidden practices that support Dedekind's elaboration of the theory of *Dualgruppe*.

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A multiple perspective approach to history of mathematics: Interplay between production of mathematics and the historical conditions of its production.

This presentation is inspired by two of the concerns discussed in Schubring (2001): historiography and the importance of the interplay between production of mathematics and the historical conditions of its production. I will present a 'multiple perspective' approach to history due to the Danish historian Eric Bernard Jensen. It will be discussed how such an approach can be adapted to history of mathematics, and its usefulness in order to unfold and discuss the interplay between production of mathematics and the historical conditions of its production (both internally to mathematics and externally given conditions). These issues will be explored through one or two concrete episodes from the history of 20th century applied mathematics: the development of mathematical programming and/or the beginning of mathematical biology. The significance of teaching and education in these episodes will also be touched upon.

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Not in the Same River Twice: On the Applicability of Mathematics in Physics

The relationship between mathematics and physics is not static but dynamic. With the development of abstract pure mathematics in the 20th century and late 19th century, this relationship came to be characterized as a "mystery", a miracle "which we neither understand nor deserve"(Wigner 1960). The miracle was understood as the unreasonable effectiveness of mathematics in the natural sciences: mathematics, the formal game and physics, the study of nature. It was with the rise of interest in pure abstract mathematics that the number theorist G. H Hardy famously "apologized" on behalf of pure mathematicians for having formal beauty (and not applicability) as their sole concern in developing mathematics. Hardy's statement (book) was then a fitting confession. Yet the triumph over the miracle of applicability soon turned into the lament over missed opportunities. The physicist Freeman Dyson in 1972 argued that the relationship between mathematics and physics has turned sour mostly due to the lack of interest from the pure mathematics' side. My paper focuses on the relationship between mathematics and theoretical physics in the aftermath of Dyson's "Missed Opportunities". I argue that from the last quarter of the 20th century (up to now) we are in a new phase of this relationship. In this phase, mathematics is inspired by the work in physics especially in superstring theory (e.g. in work of Ed Witten). Studying this newly-formed relationship, gives us insight into the characteristics of our current mathematics, which Jaffe and Quinn in 1993 called "theoretical mathematics". Moreover, the current trend toward unification at least in some areas of mathematics and theoretical physics gives us an interesting case sturdy for understanding the changing nature of mathematics, physics and their relationship. My work differs from the current philosophical literature on the applicability problem in its emphasis on the dynamic aspect of the relationship between mathematics and physics. Most philosophical works in this area have focused on rejecting Wigner's conclusion based on examples of the reasonable effectiveness of mathematics in the past. Interesting as these works might be, they make the unjustified assumption that the nature of mathematics and its relationship with physics is static (or that there is only one applicability problem). In this paper, I aim to add a historical dimension to Wigner's problem and its solutions, and to characterize both physics and mathematics as live disciplines or as Lakatos put it, research programs, that are constantly changing. It is the dynamic nature of this relation and its relata that provides a rich and complex philosophical problem for the advancement of mathematical practice.

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Material and social conditions for the development of mathematics

Since the late 1980s, science and technology studies have increasingly sought to include materiality and the actions of non-human actors in the accounts of the production of scientific knowledge (e.g., [1], [2], [3]. Although the materiality of mathematics has to some extend been included in this effort the understanding of the interplay between materiality and mathematics has mainly focused on communicative and teaching situations and we do not know much about the role of materiality in other parts of mathematical practice. Especially, we have little empirical knowledge about the role played by materiality when mathematicians work behind the closed doors of their offices. To fill this gap, we conducted a qualitative study among research mathematicians – [4]. In short the study shows that materiality enters the research practice in several different ways. External (materiel) representations thus play a crucial part in the mathematical research practice. The mathematicians not only use representation to obtain cognitive relief (e.g. by scaffolding of short term memory), they also interact heavily with the representations both in the heuristic and in the control phase of their research. Furthermore, diagrams and other pictorial representations are deliberately used as mediators that connect mathematical content to sensorimotor experiences and thus allows the mathematicians to use daily life experiences of the material world as a resource in their mathematical work – [5]. Finally, it was clear the the choice of representation to was socially sanctioned. The individual mathematicians were not at liberty to chose what kinds of representations they wanted to use in what situations; rather this choice had to be negotiated with the (close) research community. In short then, mathematicians practice with representation is a vertex where material and social forces touches upon the mathematical research practice.

In this talk we will present our main empirical findings and discuss their implications for our understanding of the mathematical practice.

References:

[1] Latour, B.; Woolgar, S. (1986). *Laboratory life: the construction of scientific facts*. Princeton, N.J.: Princeton University Press.
[2] Pickering, A. (1995). *The Mangle of Practice: Time, Agency, and Science*. Chicago: University of Chicago Press.
[3] Barad, K. M. (2007) *Meeting the Universe Halfway: Quantum Physics and the Entanglement of Matter and Meaning*. Durham, NC: Duke University Press.
[4] Johansen, M. W.; Misfeldt, M. 2014. Når matematikere undersøger matematik: og hvilken betydning det har for undersøgende matematikundervisning. *Mona*, 2014(4), 42-59.
[5] Lakoff, G.; Núñez, R. 2000. *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics Into Being*. New York: Basic Books.

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Euclid's* Elements *in XVIIth century. Some remarks

Euclide's *Elements* constitute a common cultural reference for mathematicians and philosophers (physicists) since the end of the XVIth century till the end of the XVIIth. Among the Euclidian statements, some are not deduced from priors statements: the common notions (said axioms), the demands (said postulates) and the definitions. The authors of the XVIIth century develop, indeed, towards Euclide's *Elements* two opposite criticisms. For some, Euclid would have defined too many terms and he would have demonstrated to many proposals; for others, his error would have been on the contrary a lack of definitions and demonstrations. Academician Blondel (prefacing the Elements of Geometry of Roberval) gives an overview of the oppositions which arouses Euclide's reading: *He is not the first one who notices that Euclide's do not satisfy completely the spirit of those who examine them seriously; and although it is no proposal which is of the indisputable truth, its work nevertheless, its demonstrations and the position of its principles gave rise to several people from the old time, to find fault with it. Some people said that he posed for axiom proposals which could be demonstrated by others (Proclus). Others believed that several of ,its proposals were absolutely useless and that he had omitted one very large number of new proposals which are absolutely necessary for the geometry. [] we shall say only that Mr de Roberval has [showed one] application which we could call superstitious, to demonstrate one thousand things which the others easily admitted for principles.* To convince us of the reality and the concomitance of these two attitudes, we just have to read the treaties of Roberval, champion of the first one and Antoine Arnaud, the herald of second. Both, at the same moment, undertake the same work: reread Euclide's *Elements* and reform them. The diagnosis is the same: the work of the Alexandrine is imperfect and even unfit to form in a satisfactory way the spirits which choose to venture into the study of the geometry. Antoine Arnauld publishes a first edition of his *New Elements of Geometry* in 1667 and Roberval drafts the definitive version of its *Elements of Geometry* from 1669. For the first one, Elements turn the back on the natural order and on the clear ideas; for the second, the big weakness is the insufficient demonstrative requirement. The comparative study of their Elements of geometry shows to what extent Arnauld wish "to demonstrate less ' (than Euclide) when Roberval wants"to demonstrate more '.

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From Euclidean Geometry to knots and nets: does Manders' account of Euclidean plane geometry offer a model for the analysis of contemporary mathematical proofs?

This paper assumes the success of arguments against the view that informal mathematical proofs secure rational conviction in virtue of their relations with corresponding formal derivations. This requires an alternative account of the logic of informal mathematical proofs. This paper proposes an account of those informal proofs that appeal to perception or manipulation of diagrams and mental models of mathematical phenomena. Proofs relying on mental models can be rigorous if the mental models can be externalised as diagrammatic practice. More specifically, such proofs can be rigorous if: a) it is easy to draw a diagram that shares or otherwise indicates the structure of the mathematical object; b) the information thus displayed is not metrical; and c) it is possible to label the diagram and thereby relate it systematically to syntactic, semantic, algebraic or logical notation and inference. This argument will refer principally to analyses and case studies by Manders, Giardino, Toffoli, Feferman and Leitgeb.

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Iconicity and Visualization: some Notes from Peirce's Philosophy of Mathematics

Within the framework of the Workshop "Varieties of Visualization in mathematics", the aim of this talk is to analyze the idea of visualization arising from the notion of iconicity, as it was devised by Charles S. Peirce in his theory of signs. The notion of iconicity is central in Peirce's conception of mathematics. For example, he referred to "mathematical, that is, [...] diagrammatical, or, iconic, thought." (CP 3.429), and he wrote explicitly: "Mathematical reasoning is diagrammatic. This is as true of algebra as

of geometry." (CP 5.148). In his theory of signs, Peirce proposed many classifications of signs. According to the way signs refer to the denoted entities, signs were classified into icons, indices and symbols. Diagrams fall in the category of icons, so that mathematical knowledge is the object of a rich analysis where the idea of visualization has a special role. Geometrical figures, tables and formulas are all of iconic nature. Peirce conceived mathematical activity as the construction, manipulation and observation of icons. In the case of algebra, Peirce wrote in a famous paper from 1885 "the very idea of the art is that it presents formulae which can be manipulated, and that by observing the effects of such manipulation we find properties not to be otherwise discerned. [...] These are patterns which [...] are the icons par excellence of algebra." (Peirce, CP 3.363). *Two different aspects of iconicity* will be highlighted: the *operational* aspect and the purely *topological* one. The *first* aspect focuses on icons as *structural* representations on the basis of visual properties. In this respect a diagram is characterized as "an Icon of a set of rationally related objects" (MS 293: 11) and referred to "icon [or analytic picture]" (Peirce CP 1.275). From the analysis and transformation of signs new knowledge obtains. In the exposition the elucidation of operational iconicity due to Frederik Stjernfelt will be also discussed, stressing its cognitive importance. The *second* aspect is related to the *topological* features of iconicity, which are stronger related to the idea of visualization. Some examples from mathematical logic will be provided in order to show different diagrammatic systems that are topological equivalent but have different cognitive effects leading to differences in visualization. Finally, both aspects will be connected with the tradition of *symbolic knowledge* stemming from Leibniz. Summing up, the presentation aims at contributing to the development of conceptual framework for appreciating the various epistemic roles played by the different varieties of visualizations in mathematical practices.

Reference:

Peirce, Charles Sanders. CP. *Collected Papers*. 8 vols, vols. 1- 6 ed. by Charles Hartshorne & Paul Weiss, vols. 7-8 ed. by Arthur W. Burks. Cambridge (Mass.), Harvard University Press, 1931-1958.

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Mathematical impossibility in the social sciences. The history of Arrow's impossibility theorem and its philosophical roots

In 1951 Kenneth Arrow combined welfare economy with the theory of voting and proved the first general impossibility result about the situation. When he was awarded the Nobel Prize in 1972 the press release emphasized his impossibility result thus: "As perhaps the most important of Arrow's many contributions to Welfare theory appears his "possibility theorem", according to which it is impossible to construct a social welfare function out of individual preference functions". Arrow's book has often been mentioned as a revolutionary text, and indeed it changed the approach to social choice theory dramatically. In the talk I will discuss the history of Arrow's impossibility theorem replete with multiple discoveries, priority disputes and examples of the importance of the institutional context. In particular I shall highlight the importance of philosophy for the development of Arrow's new ideas. Indeed, Arrow's new approach and his impossibility theorem was partly a result of a critique of Bentham's utilitarianism. Since economists gradually came to reject the idea that one can define a utility function (interpersonal or even individual), the mathematics of optimization of such a function under constraints lost plausibility as a mathematical model of welfare economics. Instead, Arrow suggested a new model based on abstract axiomatically defined ordered sets. This highlights the importance for applications of the modern axiomatic method in mathematics. Moreover, this method also led Arrow to a new question about voting procedures: What can one say about a voting procedure that satisfies certain axioms? The surprising answer turned out to be: It does not exist. The history of Arrow's impossibility theorem can be traced back to Condorcet's analysis of voting procedures. For the enlightenment philosopher Condorcet, voting was a way to find the truth, and it would be a mistake if voters let their own interests influence their vote. Arrow and his followers, on the other hand, denied the existence of a (Platonic) truth and considered the voting procedure as a method to weigh personal interests (and tastes) against each other. Ironically this placed their view of democracy on the level with what Condorcet considered an ancient primitive form of democracy that had been replaced by a more enlightened version in his era. But despite the almost diametrically opposite views on what voting accomplishes Condorcet and Arrow used similar mathematical methods. Thus, the history of Arrow's impossibility theorem shows examples where a change in philosophy led to a dramatic change of a mathematical model, and another example where a major difference in philosophy had no influence on the mathematical approach.

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Visualizing and Knowing in Mathematical Practice

Visualizations in mathematical practice take a wide variety of forms and serve many different epistemic ends. They can, for example, help one to understand core aspects of complex mathematical ideas, or enable one to discern connections amongst mathematical ideas that are otherwise opaque. They can reveal new mathematical possibilities, or new proof ideas. Here the focus is on one very distinctive sort of visualization in mathematical practice: the use of specially devised systems of written signs that enable one to display the contents of mathematical ideas in mathematically tractable ways, that is, in ways enabling rigorous, rule-governed manipulations of signs, reasoning *in* the system of signs. Such a system of signs is a Leibnizian universal language, at once a *characteristica* and a *calculus ratiocinator*; familiar examples of mathematical practices employing such systems of written signs are Euclidean diagrammatic practice and constructive algebraic problem solving in the symbolic language of arithmetic and algebra. Over the course of history, such systems of signs have proved extraordinarily powerful and fecund. My interest is in two aspects of this feature of such systems. I am concerned, first, with what these sorts of visualizations, and the systems they involve, can teach us about ampliative reasoning in mathematical practice. Because they display a variety of quite distinctive sorts of steps of mathematical reasoning, such visualizations of reasoning provide a very fertile ground for the philosopher reflecting on how proofs in mathematics can extend our knowledge. The second issue to be addressed through a reflection on these visualizations in mathematics is that of the a priori character of mathematical knowledge. As it is understood here, the notion of the *a priori* in mathematics concerns not certainty, or infallibility, or incorrigibility, but self-standingness, the fact that in mathematics it is always possible, in principle, to see for oneself how a proof goes. In mathematics, that is to say, one need never rely on testimony, either that of another, or that of one's own senses. Because the systems of written signs of concern here enable one visually to set out the reasoning, they also enable another to reproduce it, to enact that very chain of reasoning. These two features of devised systems of written signs which to reason—their role in enabling philosophical understanding of ampliative deductive proof and their role in clarifying the a priori character of mathematical knowledge—can, in turn, help us to understand how the mathematician achieves and maintains cognitive control in her mathematical practice.

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From counterexamples to examples, or when pathologies become the norm

Pathological objects play an important role in mathematical understanding even though there is no precise definition of them. The goal of this talk is not to try to define them but rather to comprehend the role they play within a given mathematical theory which in this case will be Mathematical Analysis. The concept of function arose as such in the works of Leibniz and Johann Bernoulli in the late seventeenth century and it became the main object of study of this field with Euler's work. We will describe briefly how the notion of function changed dramatically in the nineteenth century and we will study how this change not only came about but also how it brought on important changes for the subject both from a mathematical and a philosophical standpoint. We claim that it was from the proof of existence of a pathological function (in fact, of many pathological functions) that the nature of the concept itself was forced to change and that this development was what truly shed light on the idea of a function, so much so that by the first half of the twentieth century pathological functions had become the rightful objects studied by Mathematical Analysis. Pathologies, it would seem, rely upon certain properties occurring only in a few instances but cease to exist when these properties are held by most objects in a given class. This case study in particular also leads us to analyse the relationship that exists between pathological objects and counterexamples in mathematics. Pathological objects are usually born as counterexamples but many counterexamples do not rely on anything that can be classified as pathological.

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Cultivating Mathematicians: Memory, learning, and cognitive apprenticeship in mathematics

In the cognitive science of memory, classical models invoking the encoding and retrieval of a stored trace have been largely replaced and revised [1]. Researchers studying and modeling memory increasingly seek to adequately capture and emphasize the dynamic, distributed, and interdependent properties exhibited by a broad and ever-increasing collection of memory phenomena [2]. This research has in turn informed research in learning and education [3]; dynamic, context-dependent, generative models of human memory provide a novel base of support for the situated learning approach championed by [4], encouraging education strategies that emphasize cognitive apprenticeship and methodological development [5]. If human memory and learning consist mainly in reconfiguring the associations and apperceptions of the learner to align with expert practices, then there are compelling reasons to situate material not just *synchronically*—focusing on embodiment, physical environment, and real-world applications—but also *diachronically*. Especially for abstract material, situated apprenticeship requires witnessing the historical development, processes, and discourses of the subject. Mathematics, in particular, has been a difficult subject to assimilate into situated approaches to education, despite notable and ongoing efforts (that tend to focus on synchronic elements of situated cognition) [6]. Superficially, mathematics can seem eminently abstract and context-free; although researchers have begun revealing the human, practice-oriented, situated components of the domain [7], much work remains. Influential research applying cognitive science to mathematics education has also focused on abstraction and on cognitive capacities for abstraction [8], rather than on the cognitive apprenticeship and intransitive learning suggested by the rejection of classical models of memory. Given the nature of human memory, we contend that inducting learners into mathematical practices means attending to the historical, contingent, material, discursive practices in the development of mathematics. Teaching the historical development of mathematics in math education has garnered various support and discussion [9]; in particular, the above considerations provide support and refinement for the approach endorsed by [10], building on the work of [11], which emphasizes meta-discursive rules and methodological development as consequences of teaching history in math education. Given that the historical development of mathematics is the ongoing, situated cultivation of mathematical concepts by expert practitioners [12], the kind of history-oriented methodological development advocated by Kjeldsen and Blomhøj just is a way of situating the learner in the practices and processes of mathematics. This broadly situated, process-attentive, diachronic approach is in accordance both with the insights of the situated learning tradition and with the lessons learned from the contemporary science and philosophy of memory. Given the dynamic, intransitive, situated nature of human memory and learning, educators should attend to the human practices and historical development of mathematics in order to apprentice learners to the craft.

References:

[1] Tulving, E. (2000). Concepts of memory. In: Tulving, E.; Craik, F. (Eds.), *The Osford handbook of memory* (pp. 33-43). Ocford, UK: Oxford Univerity Press.

[2] Smith, S. M.; Vela, E. (2001). Environmental context-dependent memory: A review and meta-analysis. *Psychonomic Bulletin & review*, 892), 203-220.

[3] Brown, P. C.; Roediger, H. L.; McDaniel, M. A. (2014). *Make it stick*. Hardard University Press.

O'Loughlin, I. (2017). Learning whitout Storing: Wittgenstein's cognitive science of learning and memory. In: *A companion to Wittgenstein on Education: Pedagogical Investigations*. Peters, M. A.; Stickey, J. (Eds.), Springer.

[4] Lave, J.; Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge university press.

[5] Collins, A., Brown, J. S., & Newman, S. E. (1988). Cognitive apprenticeship: Teaching the craft of reading, writing and mathematics. *Thinking: The Journal of Philosophy for Children*, 8(1), 2-10.

Bøyum, S. (2013). Wittgenstein, Social Views and Intransitive Learning. J*ournal of Philosophy of Education, 47(3), 491*-506.

[6] Cobb, P.; Yackel, E.; Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education*, 2–33.B

Brilliant-Mills, H. (1993). Becoming a mathematician: Building a situated definition of mathematics. *Linguistics and Education*, 5(3-4), 301-334.

Núñez, R. E.; Edwards, L. D.; Filipe Matos, J. (1999). Embodied cognition as groundigs for situatedness and context in mathematics education. *Educational studies in mathematics*, 39(1), 45-65.

Freitas, E. de; Sinclair, N. (2014). *Mathematics and the Body: Material Entanglements in the Classroom*. Cambridge University Press.

[7] Rotman, B. (2000). *Mathematics an Sign: Writing, Imagining, Counting*. Stanford University Press.

Greiffenhagen, C. (2014). The materiality of mathematics: Presenting mathematics at the blackboard. *The British Journal of Sociology*, 65: 502–528.

[8] Siegler, R. S. (2003). Implications of cognitive science research for mathematics education. *A research companion to principles and standards for school mathematics*, 219-233.

Ritter, S., Anderson, J. R., Koedinger, K. R., & Corbett, A. (2007). Cognitive Tutor: Applied research in mathematics education. *Psychonomic bulletin & review*, 14(2), 249-255.

[9] Fauvel, J. (1991). Using history in mathematics education. *For the learning of mathematics*, 11(2), **3-6.**

Fried, M. N. (2001). Can mathematics education and history of mathematics coexist?. *Science & Education*, 10(4), 391-408.

[10] Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press.

[11] Muntersbjorn, M. M. (2003). Representational innovation and mathematical ontology. *Synthese*, *134*(1), 159-180.

Netz, R. (2003). *The shaping of deduction in Greek mathematics: A study in cognitive history* (Vol. 51). Cambridge University Press.

[12] Kjeldsen, T. H., & Blomhøj, M. (2012). Beyond motivation: history as a method for learning meta-discursive rules in mathematics. *Educational Studies in Mathematics*, *80*(3), 327-349.

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Semantic information and the ampliative character of formal knowledge

We propose a reassessment of the traditional theory of semantic information (TTSI). Bar-Hillel and Carnap [1] proposed TTSI as a theory for measuring the amount of information carried by sentences. In accordance with their original motivation, our interest is in the minimum semantic information associated with the propositional content of sentences. As is well known, TTSI implies Bar-Hillel and Carnap's paradox of semantic information (BCP) by which contradictions have maximum information and, by contraposition, logical truths have null information. This is problematic since it does not make room for characterizing formal knowledge as ampliative knowledge: if logical truths were not really informative, then to discover a logical truth would not change anything in our epistemic state. Nevertheless, in practice, gaining formal knowledge is not a trivial fact. In reaction, recent work [2] recent work (e.g.) proposes abandoning TTSI in despite of the good insights on the nature of semantic information that it formalizes. In contrast, we propose a more conservative solution by just changing the semantic framework that underlies TTSI: more specifically, we propose substituting classical semantics for an alternative framework known as urn semantics [3]. Urn semantics, by characterizing a way of relativizing quantifiers for parts of a given structure, defines a non-classical notion of satisfaction that enables us to define models for some classically unsatisfiable sentences, a property that blocks BCP. Now, our strategy finds motivation in arguments by [4] among others for semantic externalism. One of the fundamental claims of semantic externalism is that there is a di↵erence between the truth conditions of a sentence and the epistemological stance on these truth conditions by someone who understands the sentence. We claim that the minimum semantic information of a sentence is related with such epistemological stance that is adequately formalized by urn semantics. We argue by exploring variations of Kripke's puzzle and Twin Earth argument involving partial ignorance of the domain of quantifiers occurring in the considered sentence. Finally, we draw some comparison with alternative ways of solving 2 the problem, especially, we compare our result with that by [5]. Even though these works have some similarities, Hintikka's work faces some problems which do not challenge our result.

References:

[1] Bar-Hillel, Yehoshua; Carnap, Rudolf. Semantic information. *The British Journal for the Philosophy of Science*, 4(14):147–157, 1953.

[2] Floridi, Luciano. Outline of a theory of strongly semantic information. *Minds and machines*, 14(2):197–221, 2004.

[3]Rantala, *Veikko*. Urn models: a new kind of non-standard model for first-order logic. In: *Game-Theoretical Semantics*, pages 347–366. Springer, 1979.

Sequoiah-Grayson, Sebastian. The scandal of deduction. *Journal of Philosophical Logic*, 37(1):67–94, 2008.

[4] Putnam, Hilary. Meaning and reference. *The journal of philosophy*, 70(19):699–711, 1973.

[5] Hintikka, Jaakko. Surface information and depth information. In: *Information and inference*, pages 263–297. Springer, 1970.

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Catholicism and mathematics in the sixteenth and seventeenth century

In his paper *Production mathématique, enseignement et communication* Gert Schubring argues that the historian of mathematics must consider the religious context of the production of mathematical knowledge. In his own words "the diverse religions gave to this knowledge different socials values and functions". At the early Modernity several priests of the Roman Church made important contributions to the growing of mathematical science, produced commentaries on the works of ancient geometers, elaborated philosophical reflexions on this science and wrote several text books for his teaching. Between them we can name Clavius, Guldin ,Cavalieri, Mersenne and Arnauld. The aim of our exposition will be to show and to discuss the functions attributed by the several religious orders and priests of the Roman Church to Mathematics. These were various: Apology of Christian Faith against atheists, aid for the interpretation of the Holy Scriptures, refutation of scepticism and deism, backing for the Aristotelian philosophy, preparation for the understanding of the Truth of Christian Faith. Especially we will discuss four works: *The dissertation on the nature of Mathematics*, written by the Jesuit Biancani, *The truth of science against the sceptics* written by father Mersenne, *The new Elements of Geometry*, work of Antoine Arnauld and finally *The Elements of Geometry*, text of the Jesuit Ignace Pardies.

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Carroll's infinite regress, mathematical understanding, and the act of diagramming

A central question in the epistemology of mathematical practice is how mathematical *understanding* drives and constrains the production of mathematical proofs. In my talk, I examine this question from the perspective of Lewis Carroll's ‹What the Tortoise Said to Achilles.› In the piece, Carroll shows how an infinite regress can be generated from the demand that all premises in a deductive inference be made explicit. In the first half of the talk, drawing from Barry Stroud's ‹Inference, Belief, and Understanding', I argue that the moral of the infinite regress bears directly on the question of mathematical understanding. Understanding a mathematical proof requires understanding how the conclusion of each inference step is necessitated by the inference step's premises, and this understanding is an *act* on the part of the mathematical reasoner that is left out of any analysis that simply depicts the proof's inference steps as logically related propositions. Accordingly, one way to progress on the question of mathematical understanding is to provide accounts of the various ways acts of inferential understanding are carried out in mathematics. In the second half of the talk, I sketch the beginnings of an account that characterizes a species of such acts as *diagrammatic*. I focus specifically on a very simple geometric inference: that a point *a* is before a point *c* on an oriented geometric line if *a* is before a third point *b* and *b* is before *c*. I explore how the act of diagramming the configuration can be thought to constitute an act of understanding, and thus can be thought to carry us across the gap that Carroll's regress opens up.

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Logic and Proofs in Euclid's Geometry

According with the analysis proposed by Paul Bernays and other scholars like Ian Muller, Euclid's geometry has to be considered as a theory of constructions, in the sense that geometrical figures are studied as constructed entities. In this sense, Euclid's geometry opposes to contemporary axiomatic theories — like Hilbert's reconstruction of Euclidean geometry — which proceed from a system of objects fixed from the outset and simply describe the relationships holding between these objects. The aim of this talk is to present a formal and logical analysis of Euclid's constructive practice as it emerges from the Elements (in particular Book I). First, it is claimed that this notion cannot be captured by standard methods of constructive logic — like the witness extraction from existential formulas — since in Euclid's *Elements* there is nothing like a fixed domain of quantification from which to start. On the contrary, it is the constructive activity itself that allows one to generate, step by step, the domain of the theory. In order to give a formal and precise analysis of this point, the second part of the talk aims at studying the proof methods used in Euclid's Elements. A reconstruction of these methods is thus proposed, according to which postulates correspond to production rules (acting on terms and) allowing one to introduce new objects starting from previously given ones. This is done by means of primitive functions that correspond to the actions of drawing a point, drawing a straight line, and drawing a circle, respectively. It is then shown that a combination of these rules corresponds to a proof allowing one to solve problems, that is, to show that certain constructions are admissible from primitive ones. Moreover, in order to demonstrate that the constructed objects posses certain specific properties, a method for taking track of the relationships between the entities used during the construction process is proposed. This method consists in labelling proofs by means of relational atoms, as well as sentences formed by combinations of these relational atoms. The language used for specifying the properties of the constructed objects will be kept as minimal as possible since the final aim is to show that, contrary to what is usually believed, the logical apparatus present in the *Elements* is in fact a basic one, weaker than intuitionistic or even classical logic.

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The Applicability of Mathematics as a Philosophical Problem. Mathematization as Exploration

What makes the applicability of mathematics a philosophical problem? We juxtapose answers along two very different lines of reasoning. The first frames the problem as an ontological one: How do mathematical objects, like numbers, relate to objects in the real world? Any solution will specify on which logical grounds mathematics can be linked to realworld science whereby mathematical and real-world-objects are taken to exist independently from each other. The second reply puts applicability into an epistemological framework. The philosophical point then is rather how the process of mathematization changes both mathematical and real-world objects. Accordingly, mathematization is about exploring the universe of (potential) objects and relationships. The central fact for the applicability of mathematics therefore is the partial independence between sense and reference. The standard conception of the problem of applicability follows the first perspective. According to Mark Steiner, Frege "completely solved the semantic and the metaphysical problems of applicability" of mathematics, by defining numbers as concept extensions (Steiner, 1998. p.23). The idea of number was Frege's fundamental concern because he believed that arithmetical statements express objective thoughts and are applicable because these thoughts are about objects described in them.
In contrast, explorative mathematization assumes objects are not yet fixed. A main reason scientists find mathematics so useful is

> "an indication of how little is known about the physical world. It is only the properties well suited to mathematical description that we have been able to uncover. (…) This position is more likely assumed by consumers of mathematics rather than by mathematicians themselves. The great number of books titles like Mathematical Modelling of Hydrodynamic Phenomena, etc., is an expression of this position" (Barrow, 1992, our translation, p. 50).

This situation suggests that there is a different conception of mathematization. A different sense that highlights the creative and actively transforming nature of mathematical investigations. In this second sense, mathematization is a process that is opening up new and (not yet) conceptually conquered territory. This is what we call the explorative type of mathematization. From the latter perspective, the problem of applicability is not a logical problem, but is rather a problem of historical and epistemological nature. Mathematics has developed quite many concepts and methods to make numerically different things comparable: number, function, vector, structure, etc. etc. We make sense of facts by putting them into a specific theoretical context. Now, A = B holds, and thereby it differs from the equation A = A, besides the identical which is indicated by the equals sign, something different as well is suggested by using the different symbols A and B. Depending to where one places the identity and the difference, one can see such an equation in two ways. Frege's classic example was that "Hesperus" is the name of the "Evening Star", while "Phosphorus" (or "Lucifer") is the name of the "Morning Star"; but it turns out that the Evening Star and the Morning Star are the same thing, the planet Venus. Frege's interpretation of identity statements is due to his belief that there must be an intimate link between sense and reference as well as that the latter must prevail. However, one can conceive of A and B as different objects and then conclude that the equation designates an equal aspect or a relation between these different things. Take as an example the discovery of the relation between electricity, magnetism, and light, which were found to be different aspects of the same thing, which we call today the electromagnetic field. The two different interpretations of A = B represent in a nutshell the differences between the foundational vs the explorative conceptions of mathematization. This paper highlights mathematization as exploration and documents how relevant this type is for the problem of applicability. We do not want to abandon the foundational type, though. In driving the development of mathematics, both types assume complementary roles. If we concentrate more on the dynamical type, it is because this type is neglected philosophy. The paper comprises two main parts. The first one deals with the philosophical transformations that made explorative mathematization possible. The most important factor is the split between sense and reference, or syntax and semantics in our considerations of nature during the Scientific Revolution of the 17th century. Our starting point is Michel Foucault's (1973) *The Order of Things (Les Mots et les Choses)* and the transition from an Aristotelian science of interpretation to the Baroque and the epoch of representation. Words and things separated during the so-called "Classical" age, which on its part ended with the advent of historicism during the 19th century. Only then the relationship between sense and reference could become an object of investigation. Their partial independence opened up new possibilities of reflection which could be explored in semiotic terms. In the second part we discuss a series of three examples that highlight the complementary nature of the foundational and exploratory types of mathematization. These examples deal with the different treatments of the continuum, with the calculus ratiocinator vs. lingua universalis distinction and modern axiomatics, and with interpreting Heisenberg's matrix mechanics as an instance of explorative mathematization. We conclude by arguing how important the explorative perspective of mathematization is for gaining an adequate picture of mathematization and for the role mathematics plays for scientific knowledge.

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Mathematical objects and mathematical practice

Platonism is usually presented as the view that there are mathematical objects and they exist independently of us as self standing objects which mathematics works on. This is a metaphysical thesis, which, has, as such, little to do with mathematical practice. What has much to do with this practice are some consequences of this option concerning the admission of some mathematical methods and the appropriatness of some mathematical problems. One of these consequences is the idea that different mathematical theories can interact so as to provide complex proofs of theorems within a theory passing through other theories. Hence a natural question is this: is there a way to defend a platonistic view, whitout being forced to accept a quite doubtful metaphysics, and also whitout arguing metaphysically? In other terms, there is a way to approach the question that traditionally Platonism answer to from the point of view of mathematical practice. I'll argue that there is, by grounding of the idea that objects are constituted along this practice: they are stable intellectual contents that we can have epistemic *de re* access to. In this sense, what is essential in the platonism view is the 'works on' part, rather than the 'exist independently' part.

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The Relevance of Logic to Mathematics

What has Logic done for mathematical practice after all? What contribution, if any, has logic made for the practice of mathematicians? The answers to this question range from *everything* to *nothing.* On one pole, given that proofs are the only type of conclusive evidence in mathematics for the truth of a mathematical statement, Logic would be fundamental to mathematical practice to the extent that it investigates one of the most basic tools of mathematical practice: the construction of proofs (not to mention the well known idea that in fact mathematics is nothing but logic!). On the other pole, one could also say that logic is of little importance (or of no importance) to mathematical practice in general, given that this practice is not affected (or very little affected) by this kind of investigation: mathematicians in general would continue to prove and to calculate without any specific knowledge of proof theory or recursion theory. Philosophy for sure has affected both logic and mathematical practice: intuitionism is a good illustration of this fact. My aim in this talk is to explore some relations between two basic mathematical techniques – proof construction and calculation – in order to indicate a possible interesting way in which logic could be relevant, if not to mathematical practice in general, at least to some parts of mathematics.

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Mathematical practice vs logical normativity: the case of set-theoretic paradoxes

Cantor discovered the paradox of the greatest cardinal and its implications around 1896/97. As Ferreirós [1] puts it: he "realized perfectly well that these paradoxes were a fatal blow to the "logical" approaches to sets favored by Frege and Dedekind". Nonetheless, the discovery of the paradox didn't perturb Cantor's research on cardinal and ordinal arithmetic. It simply led him to disprove the existence of the totality of all cardinals. Indeed, he showed that the assumption of its existence contradicts his definition of a set as a comprehension of certain objects of our intuition. As remarked in [2], Cantor "is therefore not really concerned with paradoxes and their solution, but with non-existence proofs using reductio ad absurdum arguments". Hilbert formulated his own version of Cantor's paradox between 1897 and 1900 (see [2]). Hilbert thought that his version of the paradox was particularly relevant to ordinary mathematics because it avoided references to Cantor's infinite arithmetic. The aim of this paper is to compare Cantor's and Hilbert's approaches to set-theoretic paradoxes. We argue that such a comparison can shed some light on two different attitudes toward mathematical practice, as far as foundational issues are concerned. On the one hand, we argue that Cantor's position is paradigmatic of the view that a working mathematician can embrace. From this perspective, paradoxes are not a threat to the whole theory, they simply ask for some amendments at a local level. This attitude allows to maintain unaltered the theory as long as possible and seems tightly linked to a practical approach to mathematics. Following such an approach, a mathematical theory has, as it were, an internal normative role. If set theory corresponds to an accepted practice, which has a certain success in a certain community, then the norm (e.g. for correctness or success) is determined by the current practice inside the community itself. On the other hand, we argue that Hilbert's position is paradigmatic of a logical approach to the paradoxes in the foundations of mathematics. From this point of view, paradoxes are a real threat to the whole building of mathematics because they arise by applying simple and intuitive concepts in a foundational mathematical theory. This attitude looks for a general solution to the problem of paradoxes and seems to correspond to a more abstract approach. This approach assumes that the norm is imposed "from outside" the theory. The logical norm regulates and, if necessary, amends an existing practice which is considered as imperfect (as showed by the discovery of paradoxes in Cantor's set theory). In this case the norm is absolute and it does not depend on a specific theory. In the final part of the paper, we discuss which are the consequences of our analysis for the kind of set-theoretic pluralism put forward by [3] Shapiro (2014). In particular, we discuss to what extent the possibility of weakening the underlying logic rather than restricting the problematic set-theoretic principles responsible for the paradoxes constitutes a philosophically viable alternative to the standard view in the philosophy of set theory.

References:
[1] Ferreirós, J., (2016), "The Early Development of Set Theory", The Stanford Encyclopedia of Philosophy (Fall 2016 Edition), Edward N. Zalta (ed.).
[2] Peckhaus, V. & R. Kahle, (2002), "Hilbert's Paradox", Historia Mathematica, 29 (2): 157–175.
[3] Shapiro, S. (2014). Varieties of Logic, Oxford: Oxford University Press.

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Nominalistic content and the communication problem

A neglected problem in the philosophy of mathematics is "How is it that mathematicians can happily communicate despite having different views of the nature, and even the existence, of mathematical objects?" [1], sec. 10). Call this 'the communication problem' [2]. I take it that solving the communication problem is central to any account of mathematical practice. One answer to the communication problem is that what is conveyed by a typical utterance of a mathematical sentence is a content which is acceptable regardless of one's metaphysical/ontological views. I will call such content 'nominalistic content'. Any account of nominalistic content should characterize this content and address the problem of how nominalistic content gets conveyed. I will argue that Yablo's work on subject matter [3] yields a nice characterization of nominalistic content and a neat explanation of how communication between mathematicians holding different metaphysical views takes place. I will sketch a conception of contents according to which contents are connected to the subject matters or topics they address. I will argue that typical utterances of pure and applied mathematical sentences do not address the topic **w***hether there are mathematical objects*. They either address the topic *how the concrete world is* (applied mathematics) or the topic *how numbers are* (pure mathematics). Following Lewis and Yablo, I will define a notion of orthogonality between subject matters and argue that both the subject matter *how numbers are* and the subject matter *how the concrete world is* are orthogonal to *whether there are mathematical objects*. In light of this, I submit the following answer to the communication problem: the existence of abstract mathematical objects is orthogonal to the topic addressed by typical utterances of mathematical sentences. I will argue that the aforementioned subject matters are orthogoal because mathematical objects are 'preconceived objects' [4], i.e. objects whose features are fixed by the way they are characterized. If mathematical objects are preconceived, the distribution of truth values among mathematical sentences does not depend on the existence of numbers, in line with the idea, defended by Putnam and others, that there can be mathematical objectivity without mathematical objects [5] (lecture 3).

References:
[1] Yablo, Stephen (2001). Go figure: A path through fictionalism. *Midwest Studies in Philosophy* 25 (1):72–102.
[2] Liggins, David (2014). Abstract Expressionism and the Communication Problem. *British Journal for the Philosophy of Science* 65 (3):599-620.
[3] Yablo, Stephen (2014), Aboutness, Princeton U.P. Yablo, Stephen (2010).
[4] Yablo, Stephen (2010). *Things: Papers on Objects, Events, and Properties*. Oxford University Press.
[5] Putnam, H. (2009). *Ethics without ontology*. Harvard University Press.

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Contradiction, paradox and mathematical practice

In the book *How Mathematicians Think: Using Ambiguity, Contradiction, and Paradox to Create Mathematics* (Princeton University Press, 2007) William Byers presents and defends the view that mathematics is not a body of definitions, rules and theorems but, rather, it is a practice that depends essentially on intuition and creativity, and is developed in contexts where ambiguity, contradiction and vagueness have an important role. We agree with Byers' view on mathematics. We endorse the thesis that mathematics is mathematical practice, and the latter is much more liberal than the current, somewhat commonsensical, concept of mathematics as an almost algorithmic activity, that proceeds methodically, step by step, according to rules that are classical logical rules. We do not think, however, that all the aspects of mathematical activity emphasized by Byers should be called 'non-logical'. We need to free ourselves from the bounds established by classical logic and acknowledge that rationality is more than a theory of preservation of truth. In fact, rationality, and logic as an account of rationality, does not mean the unconditional avoidance of contradictions -- actually quite the contrary. The aim of this paper is to investigate to what extent a paraconsistent and paracomplete logic may be the underlying logic of the framework in which mathematical creativity arises. (Joint work with Walter Carnielli)

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Illocutionary Acts in Mathematics

Contemporary theory of illocutionary acts was originally developed inspired by Austin (1962) and further elaborated by Searle (1969, 1975, 1979) as an account of typically (and apparently purely) linguistic phenomenon, namely, the illocutionary aspects of utterances produced in the concrete use of language. In particular, this theory searched for a foundational account of the possibility of promises, orders, statements, suggestions, etc., and of the differences between these different acts. It was originally thought as a theory belonging only to the pragmatics of language and devoted solely to linguistic aspects of human actions. Later, however, this theory found widespread application in the philosophy of mind, philosophy of law and, more recently, in the foundation of social sciences, since most of the social ontology (institutions) can be seen as the product of some special illocutionary acts. The working hypothesis of this paper is that, although mathematics is usually seen as the realm of objective truths and truth-functional propositions, there are some basic and unavoidable illocutionary ingredients in the mathematical practice: if we consider that mathematical theories are produced by speakers and addressed to other speakers, they must have some illocutionary aspects. For instance, they must contain some initial stipulations (definitions, postulates, choice of vocabulary, rules of inference, etc.), and include in its metalanguage a typically performative vocabulary ('therefore', 'we conclude', etc.). These illocutionary acts create a network of what Searle calls "institutional facts" (i.e., non-natural facts) that do not belong originally to the mathematical realm, but interact with that realm and are used as a kind of platform for the study of that realm. This paper will study some historic cases and try to draw some preliminary conclusions about the interaction between these two realms.

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The role of notations in practices of 19th century logic

In his recent account of mathematical practices, Ferreirós [1] has included ideograms and notations as part of the symbolic frameworks that constitute a practice. One particular role that notations can play in the history of mathematics is explored in this talk. In particular, I will focus on two main strands in 19th-century logic: [2] *The Algebra of Logic of Boole* and his followers (e. g., Jevons and Schröder) and Frege's *Begriffsschrift*. When looking at the historical development, we can observe that although the

conceptions of logic of both Boole (and his followers) and of Frege changed quite substantially, their notations remained essentially the same, apart from some relatively minor modifications. On the one hand, the similarity of the Boolean notation to that of algebra, with which readers were familiar, was attractive and was retained also after Boole's arithmetic interpretation was rejected by Jevons and Schröder. On the other hand, the unfamiliar design of Frege's *Begriffsschrift* deterred many readers, despite it being more powerful and rigorous. Moreover, in the ensuing debates between the proponents of different conceptions (e. g., Jevons' and Schröder's criticisms of Boole, as well as the reviews of Frege's *Begriffsschrift* and his replies to them), the form of the notations were a major issue of contention and much less so their expressive power. A careful investigation and discussion of these historical developments leads to the conclusion that the notations themselves played an important role in individuating and consolidating these logical practices.

References:

[1] Ferreirós, José. 2015. *Mathematical Knowledge and the Interplay of Practices*. Princeton: Princeton University Press.

[2] Boole, George. 1847. *The Mathematical Analysis of Logic*. Cambridge: Macmillan, Barclay, and Macmillan.

______. 1854. *An Investigation of the Laws of Thought, on which are founded the mathematical theories of logic and probabilities*. London: Walton and Maberly. Reprinted by Dover Publications, New York.

Frege, Gottlob. 1879. *Begriffsschrift. Eine der arithmetischen nachgebildete Formelsprache des reinen Denkens.* Halle a/S.: Verlag Louis Nebert. English translation: *Begriffsschrift, A formula Language, Modeled upon that for Arithmetic*, in van Heijenoort (1967), pp. 1–82.

______. 1880/81. "Booles rechnende Logik und die Begriffsschrift." Published in (Frege 1969, 9–52). Translated as 'Boole's logical calculus and the concept script' in (Frege 1979, 9–46).

______. 1882. "Booles logische Formelsprache und meine Begriffsschrift." Published in (Frege 1969, 53–59). Translated as 'Boole's logical formula-language and my concept script' in (Frege 1979, 47–52).

______. 1969. *Nachgelassene Schriften und Wissenschaftlicher Briefwechsel*, vol. 1: Nachgelassene Schriften. Hamburg: Felix Meiner. Edited by Hans Hermes, Friedrich Kambartel, and Friedrich Kaulbach.

______. 1979. *Posthumous Writings*. Chicago: University of Chicago Press.

Jevons, W. Stanley. 1864. *Pure Logic or The Logic of Quality apart from Quantity with Remarks on Boole's System and on the Relation of Logic and Mathematics.* London: Edward Stanford.

Schröder, Ernst. 1877. *Der Operationskreis des Logikkalkuls*. Leipzig: B.G. Teubner. Reprint Darmstadt, 1966.

van Heijenoort, Jean. 1967. From Frege to Gödel: *A Source Book in Mathematical Logic*, 1879–1931. Cambridge, Massachusetts: Harvard University Press.
Vilkko, Risto. 1998. "The reception of Frege's *Begriffsschrift*." *Historia Mathematica* 25, pp. 412–422.

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The impact of mathematics teaching upon the development of mathematical practices

Traditionally, the teaching of mathematics has been seen as having no influence on mathematical practices and their development. The contents of teaching are seen as a certain kind of projection of academic mathematics, as a certain sedimentation. Therefore, the relation between the development of mathematical practices and the teaching of mathematics uses to be conceived of in a uni-lateral direction, without an impact of teaching upon research. Willem Kuyk, however, had denounced this traditional view in 1978, in saying: the teaching of mathematics is not a stalagmite just receiving drops from a stalactite. In this paper, as in the entire workshop it will be emphasized that there are productive interactions between the two poles. Three examples even for an impact of teaching upon mathematical practices will be presented and discussed. The first one relies on Christine Proust's research on the mathematical practices of the scribes in the Old Babylonian culture. While teaching the apprentices, the masters perfected and developed the practices of arithmetic and geometry already established. The second and the third example concern the 19^{th} century of modern times. In 1885, when Georg Cantor was still perfecting his set theory providing new fundaments for mathematics, Friedrich Meyer – friend of Cantor and mathematics teacher at the Gymnasium in Halle – elaborated a schoolbook on arithmetic and algebra, as reorganised from this basis of set theory. The third example concerns non-Euclidean geometry: In 1874, shortly after the first establishments of mathematical practices with the new geometries still meeting strong resistance from many mathematicians and from philosophers, a mathematics teacher at a Hamburg Gymnasium had published a geometry textbook according to Bolyai's notion of absolute geometry.

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Visual aspects of scientific models: the case of turbulence

Recent discussions question the role of sensory aspect in not only in proofs but also in models and thought experiments. How important are images?

Are they necessary? Relevant philosophical opinions divide into camps. For example, Brown, Gendler, Nersessian argue that visualisations are essential. Norton claims that visualisations are irrelevant in thought experiments. Meanwhile, Salis & Frigg (forthcoming) suggest that images are sometimes useful for thought experiments but never necessary. My position is that in some cases visualisations are necessary, e.g. when reasoning requires mental manipulations over them of images from diagrams (most recently Giaquinto & Starikova forthcoming, Starikova 2016, De Toffoli & Giardino 2014, 2016).
This paper moves focus from pure to applied mathematics, and to the use of images in studying physical phenomena. There are still phenomena waiting for a better mathematical grip, for example, turbulence. I will argue that a visual image (of a model of physical phenomena) can play an important role in guiding the mathematicians' research and choosing new mathematical resources. In particular, I will show that Richardson's model of a cascading wave motivated both Kolmogorov's statistical theory of turbulence and more recent geometric interpretation of the shape dynamics of a fluid volume. This is how an application of Riemannian geometry (Ricci flows) in mathematical description of turbulence became accessible. The paper distinguishes "loose" geometry, which means simply visualising a phenomenon, and "strict" geometry, which means already looking at the visual representation geometrically and applying geometry to the initial problem. On the basis of this distinction one can observe from the case study that loose geometry opens up possibilities for strict geometry. Visual representations can (even in very complex mathematics) guide research in a certain (geometric) direction, when a merely linguistic / symbolic representation does not help.

References:

Brown, James R. (1991) The Laboratory of the Mind. Cambridge: Cambridge University Press.
Gendler, Tamar S. 2004. Thought Experiments Rethought – And Reperceived. Philosophy of Science 71: 1154-1163.
Nersessian, Nancy J. 1992. "In the Theoretician's Laboratory: Thought Experimenting as Mental Modeling". Philosophy of Science 2: 291-301
Norton, John. 2004. "On thought experiments: Is there more to the argument?" Philosophy of Science 71: 1139-1151.
Salis, Fiora & Frigg, Roman (forthcoming). Capturing the scientific imagination. In Peter Godfrey-Smith & Arnon Levy (eds.), The Scientific Imagination. Oxford University Press.
Starikova, I. & Giaquinto, M. "Thought Experiment in Mathematics" in The Routledge Companion to Thought Experiments by James Robert Brown, ISBN0415735084/ 978-0415735087, [forthcoming], available online https://www.researchgate.net/publication/289813078_THOUGHT_EXPERIMENTS_IN_MATHEMATICS.

Starikova, I. 2012 “From Practice to New Concepts: Geometric Properties of Groups”, Philosophia Scientiae, 16 (1), pp.129-151, available online available online https://philosophiascientiae.revues.org/723
Starikova, I. 2011 A Philosophical Investigation of Geometrisation in Mathematics, Ph.D. thesis, University of Bristol.
Starikova, I. 2010 “Why Do Mathematicians Need Different Ways to Present Mathematical Objects? The Case of Cayley Graphs”, Topoi 29 (1), 41-51.
De Toffoli, S. & Giardino, V. (2014), ‘An Inquiry into the Practice of Proving in Low-Dimensional Topology’. Boston Studies in the History and the Philosophy of Science, 308, 315 – 336.
De Toffoli, S. & Giardino, V. (2016), ‘Envisioning Transformations. The Practice of Topology’. In: Mathematical Cultures, edited by B. Larvor, Birkhäuser Science, Springer.

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Carl Snell ‘My Revered Teacher’: Education, Euclid and System in Frege and his Environment

This paper is an installment in a larger project of reconstructing Frege’s intellectual environment and reading some of his more puzzling remarks in that context. Here I’ll discuss Frege’s view of the importance of system as displayed in Euclid’s Elements and the connections to explanation and to extending knowledge/“fruitful concepts”. The reconstructed background involves the man Frege calls his “revered teacher” Carl Snell, professor of mathematics and physics at Jena, particularly Snell’s influential writings on mathematics education. and the mathematics textbooks incorporating that approach. Snell’s approach was shaped by the “genetic” (aka “heuristic” aka “analytic”) approach of the education theorists Pestalozzi and Herbart, in opposition to the “dogmatic” (aka “synthetic”) method taken to be embodied in Euclid. Snell’s approach to pedagogy involves an explicit opposition to Euclid’s Elements as a teaching tool. Euclid, in Snell’s view, is structured artificially rather than “organically” in a way that reflects the genesis of ideas. Snell asserts that his approach produces a system with “organic structuring” (organische Gliederung), and hence it introduces the topic in a way that facilitates active learning through following the path of discovery. The phrase “organische Gliederung” was picked up and repeated in the pedagogical literature, in essays lauding Snell’s approach. (The paper will also consider the broader community of Herbartian education theorists at Jena, including others such as Leo Sachse that Frege knew well.) Frege’s remarks on the importance of “more organic” fruitful concepts for extending knowledge can be seen as an attempt to preserve central themes in Snell’s

anti-Euclidian picture of the growth of knowledge while holding on to some insights Frege attributes to Euclid.

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Looking for an account of phenomena in Math education reasoning as it occurs

The experimental evidence accumulated in the last decades in the psychological literature on reasoning shows us that the interpretation and use of the logical connectives in different contexts is far from obvious and that this is a phenomenon not limited to purely theoretical subtle concerns. This is the case, in particular, with the meaning of conditional statements. One of the effects more widely reported and studied is that these are interpreted in different ways, very commonly in disagreement with the meaning of the material implication. Nowadays standard experiments like the Wason Selection task and experiments on the use of deductive schemata give us a complex vision that can be seen (and has been very widely seen) as a lack of the normative logical competency and even as supporting our lack of rationality. It is also possible, however, to take into consideration different logical standards and accounts others than classical logic, which may not only help us to reevaluate such views, but to give us also a deeper understanding and an explanation of why we reason as we do. The same discourse is valid in regard to the Mathematical Education literature. The research in this field has been dominated by the piagetian idea that an adequate and complete description for students' stage of development is provided by an analysis of the 16 possible propositional classical connectives, together with the interaction of propositional connectives with the usual quantifiers. This is taken for granted in widely influential studies, some of them from very recent years. As an example of this situation we can consider how the tendency (present in basically all the educational levels) to treat conditional statements as biconditionals is understood. Some authors refer to this as the use of "child logic" (in contraposition to a "math logic"). This characterisation shows us sharply that this kind of phenomenon is widely interpreted only as a lack (according to the equation "a child=an incomplete adult"), and not as a phenomenon that can be characterised, explained, and even justified on its own. The prominency of this kind of common "mistakes", in fact, calls for a description of what are the processes so commonly at play, and this could also lead us eventually to an explanation of why is it so. My purpose is therefore: (1) to examine both the empiric evidence (some of it form my own experiments) related to some of the tendencies already mentioned, in a critical dialogue with the Math education literature, and (2) to give an account of these common tendencies using the tools of logic (in particular, some non-monotonic Logic Programming systems) in order to describe and understand them from other standards, different (and more adequate, I will argue) from the current ones.

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Proofs without foundations

The suggested talk will reconsider the role of foundations in our practice of mathematical proofs. We are used to thinking of proofs as procedures that guarantee the correctness of mathematical claims. We further tend to assume that in order to do that, proofs must advance from well-established foundations to less established conclusions. Of course, it is well established that proofs have a host of other functions, such as: to convince, to explain *why* a claim is true, to systematize knowledge, to discover new truths, to set a standard for mathematical communication, to demonstrate the use of auxiliary mathematical notions, and to exercise our capacity to reason. However, an argument is likely to have its status of “proof” revoked, if it fails to lead us with sufficient certainty form established foundations to a stated conclusion. In my presentation I will argue for the historical and cognitive possibility of cultures of proofs without foundations (I do not refer here only to foundations in the modern rigorous sense, but also to a much broader, traditional sense of the term “foundations”). The historical support will come from Śaṅkara’s 16th century *Kriyaakramakarii* – a Sanskrit commentary to Bhaskara’s 12th century canonical treatise *Liilaavatii*. I will argue that this commentary, which makes an effort to provide many mathematical proofs, establishes no clear starting points or directionality for proofs, and therefore no mathematical foundations. Instead, the function of proofs in this commentary is to relate different kind of mathematical knowledge into an integrated system. The cognitive support for the possibility of cultures of proofs without foundations will come from non-modular models of embodied cognition. A typical understanding of embodied mathematical cognition hypothesizes a common embodied experienced foundation, which serves as grounds for more abstract mathematical notions. I will present alternative cognitive models to argue that even abstract notions feed back to reshape our embodied experience. This means that the role of proof is to cognitively relate notions in a state of flux, rather than proceed from established foundation to tentative abstraction. This understanding can be related to Wittgenstein’s and Lakatos’s views on the role of mathematical proofs.

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Diagrams and formulas: On the contents of representations in mathematics

According to the usual contemporary definition, a mathematical proof is at bottom a sequence of sentences, each of which follows from the preceding ones or is an axiom or assumption. In some mathematical practices, however, proofs contain elements other than sentences, and so do not easily fit this definition. The proofs in Euclid's *Element*s, for instance, often make essential use of geometrical diagrams (see [1]). Can we still consider them proofs? How did Euclid use the diagrams to ensure reliable reasoning? Motivated by epistemological questions of this kind, many scholars associated with the philosophy of mathematical practice have studied diagrammatic reasoning in recent years (see for instance the papers in [2]). My goal in this presentation would be to argue that the use of symbolic formulas in proofs can raise some of the same issues as the use of diagrams: formulas, or symbolic expressions more generally, cannot unproblematically be treated like sentences in natural language. Indeed, we will see that formulas are sometimes used not so much in order to assert specific claims ("such-and-such a complex relation holds between quantities x,y,z"), but rather as displays from which various pieces of information may be extracted: for instance, I may derive and write down a complex formula linking x and y only in order to observe that x is given by a second-degree polynomial in y. In other words, formulas are not just asserted; they are inspected, much like diagrams are. To substantiate this claims, I will examine the use of formulas in several episodes of the early history of the calculus of operations, in the late seventeenth and eighteenth centuries. By calculus of operations, I refer to the manipulation of operators, for instance the "d" of differentiation, as algebraic quantities whose powers correspond to iterated applications of the operator. The earliest incarnation of this idea is Leibniz's "analogy between powers and differences", developed in 1695 in correspondence with Johann Bernoulli. Leibniz and Bernoulli's idea will be taken up again by Lagrange; various authors will then try, in the last decades of the eighteenth century, to put these methods on a solid footing, first and foremost Laplace.

References:

[1] Manders, Ken (2008). "The Euclidean Diagram (1995)". In: *The Philosophy of mathematical practice*, ed. by Paolo Mancosu, Oxford: Oxford University Press, pp. 80-133.
[2] Mumma, John; Panza, Marco; Sandu, Gabriel (2012). *Diagrams in mathematics: History and Philosophy*. Special issue of Synthese, vol. 186(1).

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